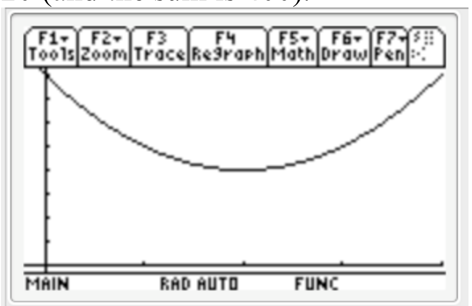
Calculus Section 4.4 Modeling and Optimization

1) a) The domain of this problem is [0,20]

$x + y = 20 \Rightarrow y = 20 - x \quad S = x^2 + (20 - x)^2 = 2x^2 - 40x + 400$

$S' = 4x - 40 \quad 4x - 40 = 0 \Rightarrow x = 10 \quad S'' = 4$

By the second derivative test, we have a minimum value for the sum of the squares when the two numbers are 10 and 10 (the sum is 200). We have a maximum at the endpoints of the domain, where the numbers are 0 and 20 (and the sum is 400).



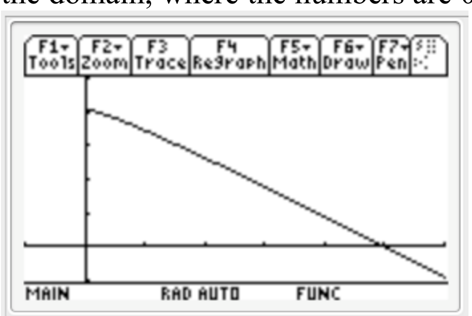
[-1,20] by [-10,410]

1) b) The domain of this problem is [0,20]

$x + y = 20 \Rightarrow y = 20 - x \quad S = \sqrt{x} + 20 - x$

$S' = \frac{1}{2\sqrt{x}} - 1 = \frac{1 - 2\sqrt{x}}{2\sqrt{x}} \quad 1 - 2\sqrt{x} = 0 \Rightarrow x = \frac{1}{4} \quad S'' = -\frac{1}{4\sqrt{x^3}}$

By the second derivative test, we have a maximum value for the sum of the square root of one number and the other occurs when the two numbers are 1/4 and 79/4 (the sum is 81/4). We have a minimum at the endpoints of the domain, where the numbers are 0 and 20 (and the sum is $\sqrt{20}$).



[-5,30] by [-5,25]

2) Domain of the function is (0,5)

$$x^2 + y^2 = 25 \Rightarrow y = \sqrt{25 - x^2}$$

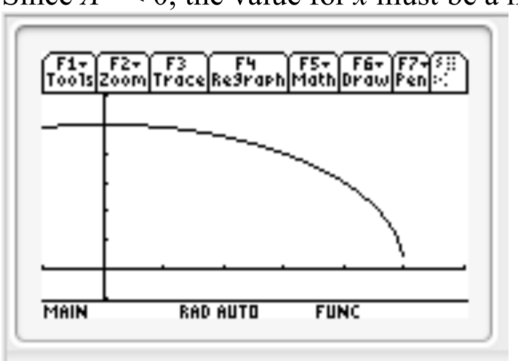
$$A = \frac{1}{2}x\sqrt{25 - x^2} \quad A' = \frac{1}{2}x\left(\frac{1}{2}\left(25 - x^2\right)^{-1/2}\left(-2x\right)\right) + \frac{1}{2}\sqrt{25 - x^2} = \frac{-x^2 + 25 - x^2}{2\sqrt{25 - x^2}} = \frac{25 - 2x^2}{2\sqrt{25 - x^2}}$$

$$25 - 2x^2 = 0 \Rightarrow x = \sqrt{\frac{25}{2}} = \frac{\sqrt{50}}{2} \approx 3.536 \quad y = \sqrt{\frac{25}{2}} \approx 3.536$$

$$A'' = \frac{2\sqrt{25 - x^2}\left(-4x\right) - \left(25 - 2x^2\right)\left(\left(25 - x^2\right)^{-1/2}\left(-2x\right)\right)}{4\left(25 - x^2\right)} = \frac{-8x\left(25 - x^2\right) + 4x\left(25 - x^2\right)}{4\left(25 - x^2\right)^{3/2}} = \frac{x^3 - 25x}{\left(25 - x^2\right)^{3/2}}$$

$$A''\left(\sqrt{\frac{25}{2}}\right) = \frac{\frac{25}{2}\sqrt{\frac{25}{2}} - 25\sqrt{\frac{25}{2}}}{\left(25 - \frac{25}{2}\right)^{3/2}} < 0$$

Since $A''<0$, the value for x must be a maximum by the 2nd derivative test.



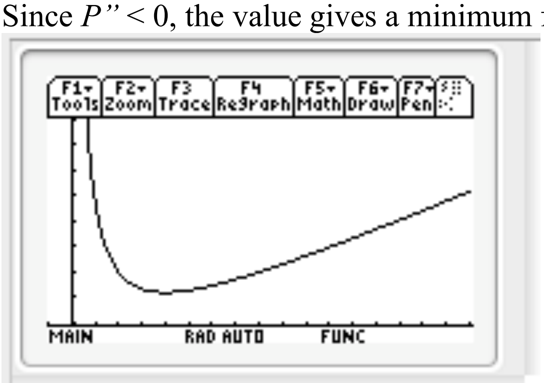
[-1,6] by [-1,6]

3)

$$lw = 16 \Rightarrow w = \frac{16}{l} \quad P = 2l + 2w = 2l + \frac{32}{l} \quad P' = 2 - \frac{32}{l^2} \quad P'' = \frac{64}{l^3}$$

$$2 - \frac{32}{l^2} = 0 \Rightarrow l^2 = 16 \Rightarrow l = 4 \quad P''(4) > 0$$

Since $P''<0$, the value gives a minimum for the perimeter by the 2nd derivative test.



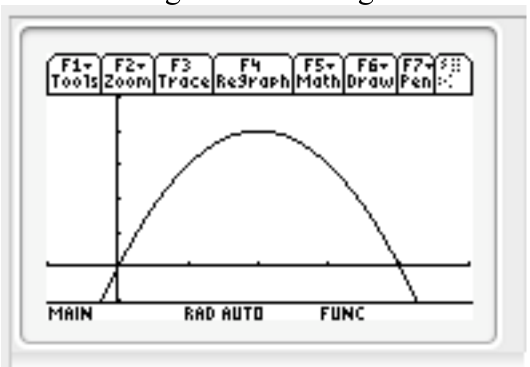
[-1,17] by [10,50]

4) Domain of the function is (0,4)

$2l + 2w = 8 \Rightarrow w = 4 - l \quad A = l(4-l) = 4l - l^2 \quad A' = 4 - 2l \quad A'' = -2$

$4 - 2l = 0 \Rightarrow l = 2 \Rightarrow w = 2$

Since the 2nd derivative is less than zero, $l = w = 2$ is a maximum value, so the square with sides of 2 m would be the rectangle with the largest area of all rectangles with a perimeter of 8 m.



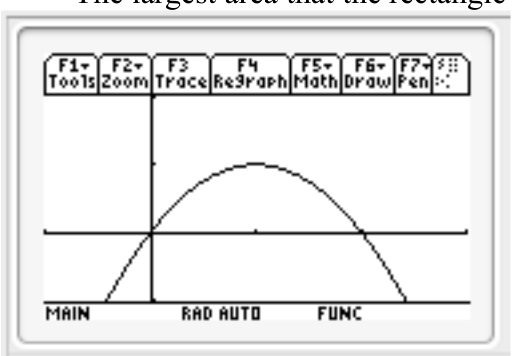
[-1,5] by [-1,5]

5) a) $P(x, 1-x)$

b) $A = 2x(1-x) = 2x - 2x^2$

c) $A' = 2 - 4x \quad A'' = -4 \quad 2 - 4x = 0 \Rightarrow x = \frac{1}{2}$. By the 2nd derivative test, this value must be a maximum.

The largest area that the rectangle can have is ½, the dimensions of the rectangle are 1 unit by ½ unit.



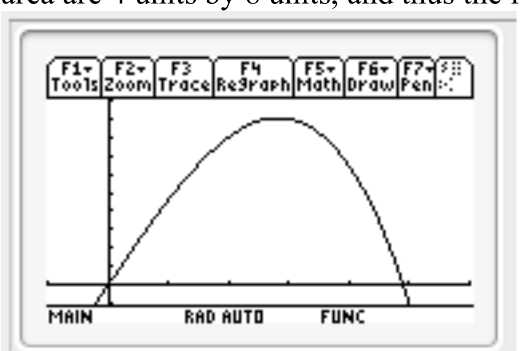
[-1,3] by [-1,2]

6)

$A = 2x(12 - x^2) = 24x - 2x^3 \quad A' = 24 - 6x^2 \quad A'' = -12x$

$24 - 6x^2 = 0 \Rightarrow x = 2$

The 2nd derivative test shows that this value must be a maximum. The dimensions of the rectangle of greatest area are 4 units by 8 units, and thus the largest area is 32 square units.



[-1,6] by [-5,50]

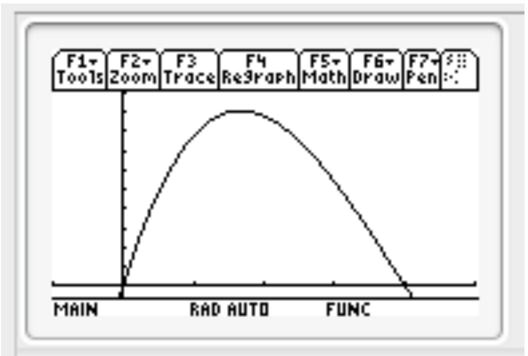
7)

$$v = (8-2x)(15-2x)x = 120x - 46x^2 + 4x^3 \quad A' = 120 - 92x + 12x^2 \quad A'' = -92 + 24x$$

$$12x^2 - 92x + 120 = 0 \Rightarrow 4(3x^2 - 23x + 30) = 0 \Rightarrow 4(3x-5)(x-6) \Rightarrow x = 6, \frac{5}{3}$$

6 is an impossible value for *x*. *A''* < 0 for $x = \frac{5}{3}$, so by the 2nd derivative test this must be a maximum value.

The dimensions for the maximum volume would be $\frac{5}{3}$ in $\times \frac{14}{3}$ in $\times \frac{35}{3}$ in , and the maximum volume is $\frac{2450}{27}$ in^3



[-1,5] by [-5,100]

8)

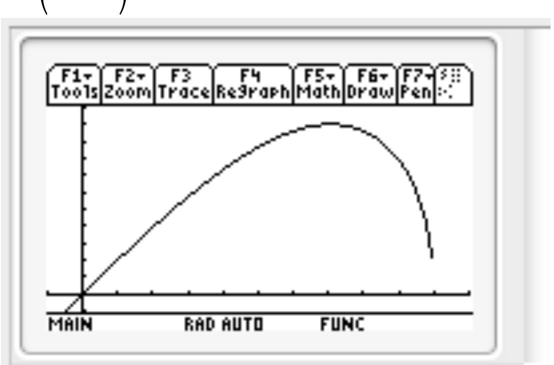
$$a^2 + b^2 = 20^2 \Rightarrow b = \sqrt{400 - a^2}$$

$$A = \frac{1}{2}a\sqrt{400-a^2} \quad A' = \frac{1}{2}a\left(\frac{1}{2}(400-a^2)^{-1/2}(-2a)\right) + \frac{1}{2}\sqrt{400-a^2} = \frac{-a^2 + 400 - a^2}{2\sqrt{400-a^2}} = \frac{400 - 2a^2}{2\sqrt{400-a^2}}$$

$$400 - 2a^2 = 0 \Rightarrow a = \sqrt{\frac{400}{2}} = 10\sqrt{2} \approx 14.142 \quad b = \sqrt{\frac{400}{2}} = 10\sqrt{2} \approx 14.142$$

$$A'' = \frac{2\sqrt{400-a^2}(-4a) - (400-2a^2)\left((400-a^2)^{-1/2}(-2a)\right)}{4(400-a^2)} = \frac{-8a(400-a^2) + 4a(400-a^2)}{4(400-a^2)^{3/2}} = \frac{a^3 - 400a}{(400-a^2)^{3/2}}$$

$$A''(10\sqrt{2}) < 0$$



[-2,22] by [-10,110]

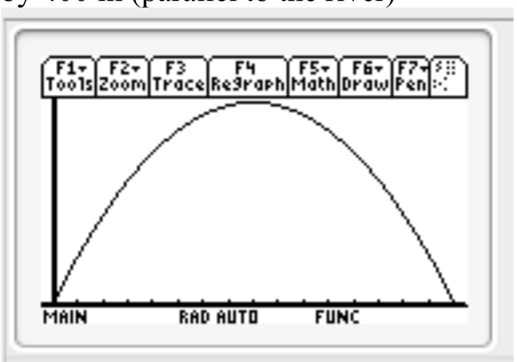
9)

$$2x + y = 800 \Rightarrow y = 800 - 2x$$

$$A = xy = x(800 - 2x) = 800x - 2x^2 \quad A' = 800 - 4x \quad A'' = -4$$

$$800 - 4x = 0 \Rightarrow x = 200 \quad y = 800 - 400 = 400$$

By the 2nd derivative test, this value is a maximum.
The largest area that can be enclosed is 80,000 m^2, with dimensions 200 m (perpendicular to the river) by 400 m (parallel to the river)



[-10,410] by [-10,81000]

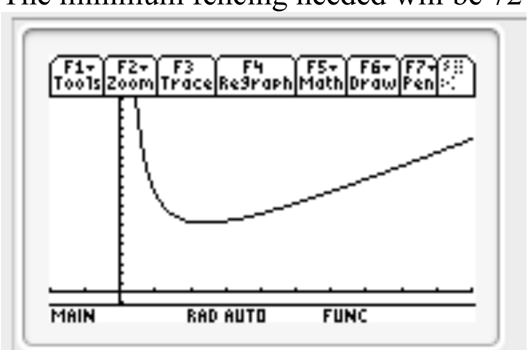
10)

$$xy = 216 \Rightarrow y = \frac{216}{x}$$

$$P = 3x + 2y = 3x + \frac{432}{x} \quad P' = 3 - \frac{432}{x^2} \quad P'' = \frac{864}{x^3}$$

$$3 - \frac{432}{x^2} = 0 \Rightarrow x^2 = 144 \Rightarrow x = 12 \quad y = \frac{216}{12} = 18$$

By the 2nd derivative test, this value is a minimum.
The minimum fencing needed will be 72 m, making a 12 m by 18 m rectangle with a 12 m divider.



[-10,50] by [-10,200]

11)

$$V = lwh = l^2h \quad 500 = l^2h \Rightarrow h = \frac{500}{l^2}$$

$$SA = l^2 + 4lh = l^2 + 4l\left(\frac{500}{l^2}\right) = l^2 + \frac{2000}{l} \quad SA' = 2l - \frac{2000}{l^2} \quad SA'' = 2 + \frac{2000}{l^3}$$

$$2l - \frac{2000}{l^2} = 0 \Rightarrow l^3 = 1000 \Rightarrow l = 10$$

By the 2nd derivative test, this value is a minimum.
The dimensions that will make the tank weigh as little as possible (least surface area) is 10 ft by 10 ft (on the base) by 5 ft high.

12)

$$V = lwh = x^2y \quad 1125 = x^2y \Rightarrow y = \frac{1125}{x^2}$$

$$c = 5x^2 + 30xy = 5x^2 + 30x\left(\frac{1125}{x^2}\right) = 5x^2 + \frac{33750}{x}$$

$$c' = 10x - \frac{33750}{x^2} \quad c'' = 10 + \frac{33750}{x^3}$$

$$10x - \frac{33750}{x^2} = 0 \Rightarrow x = 15 \quad y = \frac{1125}{15^2} = 5$$

By the 2nd derivative test, the value is a minimum
The dimensions of the tank that will minimize the total cost is 15 ft by 15 ft (the base) by a height of 5 ft.

13) Call the dimensions of the printed area x for width and y for height.

$$xy = 50 \Rightarrow y = \frac{50}{x} \quad \text{With margins: width} = x + 4,\ \text{height} = y + 8$$

$$A = (x+4)(y+8) = (x+4)\left(\frac{50}{x} + 8\right) = 50 + 8x + \frac{200}{x} + 32 = 82 + 8x + \frac{200}{x}$$

$$A' = 8 - \frac{200}{x^2} \quad A'' = \frac{400}{x^3} \quad 8 - \frac{200}{x^2} = 0 \Rightarrow x^2 = 25 \Rightarrow x = 5$$

By the 2nd derivative test, this value is a minimum.
The dimensions that minimize the amount of paper used are 9 inches wide by 18 inches high.

14) a) $v = -32t + 96 \quad v(0) = 96\,\text{ft/sec}$

b) $-32t + 96 = 0 \Rightarrow t = 3$ seconds after it starts, $s = -16(3^2) + 96(3) + 112 = 256\,\text{ft}$.

c) $-16t^2 + 96t + 112 = 0 \Rightarrow -16(t^2 - 6t - 7) = 0 \Rightarrow -16(t-7)(t+1) = 0 \Rightarrow t = -1, 7$

Disregard t = -1. At t = 7, $v(7) = -32(7) + 96 = -128$ ft/sec

15)

$$A = \frac{1}{2}ab\sin\theta \quad \frac{dA}{d\theta} = \frac{1}{2}ab\cos\theta \quad \frac{d^2A}{d\theta^2} = -\frac{1}{2}ab\sin\theta$$

$$\frac{1}{2}ab\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

By the 2nd derivative test, this value is a maximum.

16)

$$V = \pi r^2 h \quad 1000 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$$

$$SA = 2\pi rh + \pi r^2 = \frac{2000}{r} + \pi r^2 \quad SA' = 2\pi r - \frac{2000}{r^2} \quad SA'' = 2\pi + \frac{4000}{r^3}$$

$$2\pi r - \frac{2000}{r^2} = 0 \Rightarrow \pi r^3 = 1000 \Rightarrow r = \frac{10}{\sqrt[3]{\pi}} \approx 6.83$$

By the 2nd derivative test, this value will be a minimum.

17)

$$V = \pi r^2 h \Rightarrow 1000 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$$

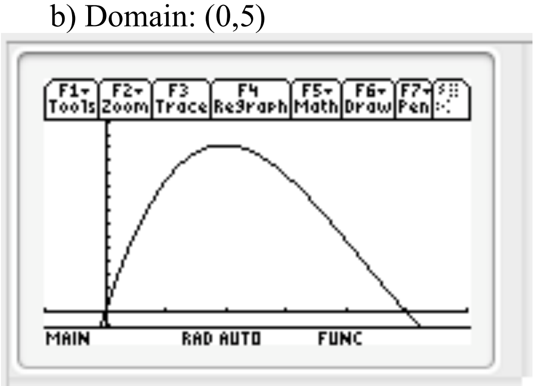
$$SA = 8r^2 + 2\pi rh = 8r^2 + \frac{2000}{r} \quad SA' = 16r - \frac{2000}{r^2} \quad SA'' = 16 + \frac{2000}{r^3}$$

$$16r - \frac{2000}{r^2} = 0 \Rightarrow 16r^3 = 2000 \Rightarrow r = 5 \quad h = \frac{1000}{\pi(25)} = \frac{40}{\pi}$$

$\frac{h}{r} = \frac{\frac{40}{\pi}}{5} = \frac{8}{\pi}$, so the ration is $\frac{8}{\pi}$ to 1.

18) a) $V = (7.5 - x)(10 - 2x)x = 75x - 25x^2 + 2x^3$

b) Domain: (0,5)



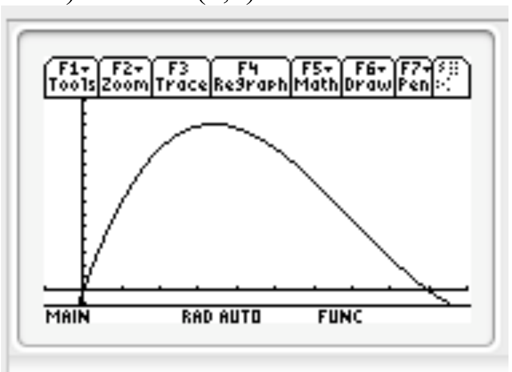
[-1,6] by [-5,75]

c) Maximum volume is 66.019 in3 when $x \approx 1.962$

d) $V' = 75 - 50x + 6x^2 \quad 75 - 50x + 6x^2 = 0 \Rightarrow x = \frac{50 \pm \sqrt{700}}{12} \approx 1.962$ or 6.371 (6.371 is out of the domain)

19) a) $V = 2(18-2x)(24-2x)x = 864x - 168x^2 + 8x^3$

b) Domain (0,9)



[-1,10] by [-100,1500]

c) 1309.632 in^3 when $x = 3.456$

d) $V' = 864 - 336x + 24x^2 \quad 864 - 336x + 24x^2 = 0 \Rightarrow x \approx 3.394, 10.606$. Disregard 10.606 (out of domain)

e) $1120 = 864x - 168x^2 + 8x^3 \Rightarrow x = 2, 5, 14$. Disregard 14 (out of domain)

20)

$$T = \frac{\sqrt{x^2+4}}{2} + \frac{6-x}{5} = \frac{1}{2}\sqrt{x^2+4} - \frac{1}{5}x + \frac{6}{5}$$

$$T' = \frac{1}{4}\left(x^2+4\right)^{-1/2}(2x) - \frac{1}{5} = \frac{x}{2\sqrt{x^2+4}} - \frac{1}{5} = \frac{5x - 2\sqrt{x^2+4}}{10\sqrt{x^2+4}}$$

$$\frac{5x-2\sqrt{x^2+4}}{10\sqrt{x^2+4}} = 0 \Rightarrow 5x - 2\sqrt{x^2+4} = 0 \Rightarrow \sqrt{x^2+4} = \frac{5x}{2} \Rightarrow x^2 + 4 = \frac{25x^2}{4} \Rightarrow 21x^2 = 16 \Rightarrow x = \frac{4}{\sqrt{21}} \approx 0.873$$

21)

$$A = 2x\left(4\cos(0.5x)\right) = 8x\cos(0.5x)$$

$$A' = 8x\left(-\sin(0.5x)\right)(0.5) + 8\cos(0.5x) = 8\cos(0.5x) - 4x\sin(0.5x)$$

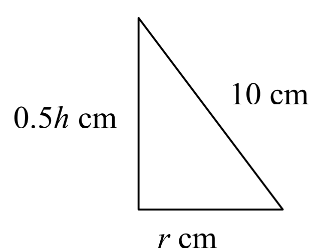
$$A'' = -8\sin(0.5x)[0.5] - 4x\cos(0.5x) - 4\sin(0.5x) = -4x\cos(0.5x) - 8\sin(0.5x)$$

$$8\cos(0.5x) - 4x\sin(0.5x) = 0 \Rightarrow x \approx 1.721$$

The 2nd derivative test shows that this value is a maximum.

The dimensions of the rectangle with the largest area are 3.442 by 2.608, the maximum area is approx. 8.977.

22)



$$r^2 + (0.5h)^2 = 10^2 \Rightarrow r^2 + 0.25h^2 = 100 \Rightarrow r^2 = 100 - 0.25h^2$$

$$V = \pi r^2 h \Rightarrow V = \pi\left(100 - 0.25h^2\right)h \Rightarrow V = 100\pi h - 0.25\pi h^3$$

$$V' = 100\pi - 0.75\pi h^2 \quad V'' = -1.5\pi h$$

$$100\pi - 0.75\pi h^2 = 0 \Rightarrow h^2 = \frac{400}{3} \Rightarrow h = \frac{20}{\sqrt{3}}$$

By the 2nd derivative test, this value will give us a maximum volume.

The maximum volume will be $100\pi\left(\frac{20}{\sqrt{3}}\right) - 0.25\pi\left(\frac{20}{\sqrt{3}}\right)^3 \approx 2418.399 \text{ cm}^3$

23)

$$p(x) = r(x) - c(x) = 8\sqrt{x} - 2x^2$$

$$p'(x) = \frac{4}{\sqrt{x}} - 4x \quad p''(x) = \frac{2}{\sqrt{x^3}} - 4$$

$$\frac{4}{\sqrt{x}} - 4x = 0 \Rightarrow \frac{4}{\sqrt{x}} = 4x \Rightarrow 4 = 4x^{3/2} \Rightarrow x^{3/2} = 1 \Rightarrow x = 1$$

By the 2nd derivative test, this value will produce a maximum.
The production level that maximizes profit is 1000 units.

24)

$$p(x) = r(x) - c(x) = \frac{x^2}{x^2+1} - \frac{(x-1)^3}{3} + \frac{1}{3}$$

$$p'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} - (x-1)^2 = \frac{2x}{(x^2+1)^2} - (x-1)^2$$

$$p''(x) = \frac{(x^2+1)^2(2) - 2x(2(x^2+1)(2x))}{(x^2+1)^4} - 2(x-1) = \frac{2}{(x^2+1)^2} - \frac{8x^2}{(x^2+1)^3} - 2(x-1)$$

$$\frac{2x}{(x^2+1)^2} - (x-1)^2 = 0 \Rightarrow x \approx 0.294, 1.525$$

$$p''(0.294) \approx 2.567 \quad p''(1.525) \approx -1.375$$

By the 2nd derivative test, a production level of 1, 375 units will maximize profits.

25) **Average** cost = cost divided by number of items produced = $c(x)/x$

$$\frac{c(x)}{x} = \frac{x^3 - 10x^2 - 30x}{x} = x^2 - 10x - 30$$

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 10 \quad 2x - 10 = 0 \Rightarrow x = 5$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2$$

By the 2nd derivative test, the average cost will be minimized at a production level of 5,000 units.

26) **Average** cost = cost divided by number of items produced = $c(x)/x$

$$\frac{c(x)}{x} = \frac{xe^x - 2x^2}{x} = e^x - 2x$$

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = e^x - 2 \quad e^x - 2 = 0 \Rightarrow x = \ln 2$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = e^x \quad e^{\ln 2} = 2$$

By the 2nd derivative test, the average cost will be minimized at a production level of ln2 th
ousand units, or approx. 693 units.

27) Profit = revenue – cost

$r(x) = x\left(200 - 2(x-50)\right) = 300x - 2x^2$

$c(x) = 6000 + 32x$

$p(x) = 300x - 2x^2 - \left(6000 + 32x\right) = -6{,}000 + 268x - 2x^2$

$p'(x) = 268 - 4x \qquad p''(x) = -4$

$268 - 4x = 0 \Rightarrow x = 67$

By the 2nd derivative test, $x = 67$ must create a maximum profit.

28) a)

$f(x) = xe^{-x} \quad f'(x) = -xe^{-x} + e^{-x} \quad f''(x) = xe^{-x} - 2e^{-x}$

$-xe^{-x} + e^{-x} = 0 \Rightarrow e^{-x}\left(-x+1\right) = 0 \Rightarrow x = 1$

$f''(1) = e^{-1} - 2e^{-1} = -\frac{1}{e}$

By the 2nd derivative test, $x = 1$ is the maximum value.

b)

29) a)

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c \quad f''(x) = 6ax + 2b$

$3ax^2 + 2bx + c$ will have either 2 real solutions for x (including the possibility of a "double root"), or 0 real solutions for x.

b) 1) $f(x) = 3x^3 + \frac{1}{2}x^2 + 8x - 2$ has 0 local extrema, as $f'(x) = 6x^2 + x + 8$, and

$6x^2 - x + 8 = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 - 4(6)(8)}}{12}$, so the solutions are not real values.

b) 2) $f(x) = \frac{1}{3}x^3 - x^2 + x - 2$ has 0 (1 "double root) local extrema, as $f'(x) = x^2 - 2x + 1$, and $x^2 - 2x + 1 = 0 \Rightarrow x = 1$, but $f''(x) = 2x - 2 \Rightarrow f(1) = 0$, so there is a point of inflection at $x = 1$, not a local extremum.

b) 3) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 5$ has 2 local extrema, as $f'(x) = x^2 + x - 2$, and $x^2 + x - 2 = 0 \Rightarrow \left(x+2\right)\left(x-1\right) = 0 \Rightarrow x = 1, -2$. $f''(x) = 2x + 1 \Rightarrow f(1) = 3, f(-2) = -3$, so there is a local maximum at $x = -2$ and a local minimum at $x = 1$.

30)

$l + 4w = 108 \Rightarrow l = 108 - 4w$

$V = lwh = \left(108 - 4w\right)\left(w\right)\left(w\right) = 108w^2 - 4w^3$

$V' = 216w - 12w^2 \quad V'' = 216 - 24w$

$216w - 12w = 0 \Rightarrow w = 0, 18 \quad V''(0) = 0, V''(18) = -216$

By the second derivative test, a maximum volume is attained when the square end is 18 in. by 18 in., and the length is 36 in.

31) a)

$$2x + 2y = 36 \Rightarrow x = 18 - y \quad V = \pi r^2 h \quad C = 2\pi r \quad x = 2\pi r \Rightarrow r = \frac{x}{2\pi} = \frac{18 - y}{2\pi}$$

$$V = \pi\left(\frac{18 - y}{2\pi}\right)^2 (y) = \frac{324\pi y - 36\pi y^2 + \pi y^3}{2\pi} = 162y - 18y^2 + \frac{y^3}{2}$$

$$V' = 162 - 36y + \frac{3}{2}y^2 \quad V'' = -36 + 3y$$

$$162 - 36y + \frac{3}{2}y^2 = 0 \Rightarrow y = 6,18 \quad V''(6) = -18, V''(18) = 18$$

By the second derivative test, there will be a maximum volume when y = 6 cm and x = 12 cm.

b)

$$2x + 2y = 36 \Rightarrow y = 18 - x \quad V = \pi r^2 h = \pi\left(x^2\right)\left(18 - x\right) = 18\pi x^2 - \pi x^3$$

$$V' = 36\pi x - 3\pi x^2 \quad V'' = 36\pi - 6\pi x$$

$$36\pi x - 3\pi x^2 = 0 \Rightarrow x = 0,12 \quad V''(0) = 36\pi, V''(12) = -36\pi$$

By the second derivative test, there will be a maximum volume when y = 6 cm and x = 12 cm.

32)

$$r^2 + h^2 = \left(\sqrt{3}\right)^2 \Rightarrow r = \sqrt{3 - h^2} \quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\sqrt{3 - h^2}\right)^2 h = \pi h - \frac{1}{3}\pi h^3$$

$$V' = \pi - \pi h^2 \quad V'' = -2\pi h$$

$$\pi - \pi h^2 = 0 \Rightarrow h = 1 \quad V''(1) = -2\pi$$

By the 2nd derivative test, the maximum volume will be when the height is 1 m and the radius is $\sqrt{2}$ m.

The volume will be $V = \frac{1}{3}\pi\left(\sqrt{2}\right)^2(1) = \frac{2}{3}\pi \text{ m}^3$.

33) a)

$$f'(x) = 2x - \frac{a}{x^2} \quad f''(x) = 2 + \frac{2a}{x^3}$$

$$2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3 = 2(2)^3 = 16$$

By the 2nd derivative test, this value is a minimum.

b)

$$f'(x) = 2x - \frac{a}{x^2} \quad f''(x) = 2 + \frac{2a}{x^3}$$

$$2 + \frac{2a}{x^3} = 0 \Rightarrow 2a = -2x^3 \Rightarrow a = -x^3 = -1$$

34)

$f'(x) = 2x - \frac{a}{x^2} \quad f''(x) = 2 + \frac{2a}{x^3}$

$2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3$ These values are where local extrema will occur.

$f''(x) = 2 + \frac{2(2x^3)}{x^3} = 6$

Since the 2nd derivative value is positive for all values of *a*, there cannot be a local maximum, only a local minimum, for any value of *a*.

35) a)

$f'(x) = 3x^2 + 2ax + b \quad f''(x) = 6x + 2a$

$f'(-1) = 3 - 2a + b \quad f'(3) = 27 + 6a + b$ Both of these must occur when the derivative $=$ 0, so

$3 - 2a + b = 27 + 6a + b \Rightarrow 8a = -24 \Rightarrow a = -3 \quad 3 - 2(-3) + b = 0 \Rightarrow b = -9$

$f''(-1) = 6(-1) + 2(-3) = -12 \quad f''(3) = 6(3) + 2(-3) = 12$

The 2nd derivative confirms that these values for *a* and *b* give a maximum at $x = -1$ and a minimum at $x = 3$

b)

$f'(x) = 3x^2 + 2ax + b \quad f''(x) = 6x + 2a$

$f'(4) = 48 + 8a + b \quad f''(1) = 6 + 2a$ Both of these must occur when the derivative $=$ 0, so

$48 + 8a + b \Rightarrow \quad b = -48 - 8a \quad 6 + 2a = 0 \Rightarrow a = -3 \quad b = -48 - 8(-3) = -24$

$f''(4) = 6(4) + 2(-3) = 18$

The 2nd derivative confirms that these values for *a* and *b* give a minimum at $x = 4$.

36)

$x^2 + y^2 = 3^2 \Rightarrow x^2 = 9 - y^2$

$V = \frac{1}{3}\pi x^2(3 + y) = \frac{1}{3}\pi(9 - y^2)(3 + y) = 9\pi + 3\pi y - \pi y^2 - \frac{\pi}{3}y^3$

$V' = 3\pi - 2\pi y - \pi y^2 \quad V'' = -2\pi - 2\pi y$

$3\pi - 2\pi y - \pi y^2 = 0 \Rightarrow 3 - 2y - y^2 = 0 \Rightarrow (3 + y)(1 - y) = 0 \Rightarrow y = 1, -3$

$V''(1) = -4\pi$

$V = \frac{1}{3}\pi(9 - 1)(3 + 1) = \frac{32}{3}\pi$ cubic units

The 2nd derivative test shows that this is a maximum volume.

37) a)

$w^2 + d^2 = 12^2 \Rightarrow d^2 = 144 - w^2$

$S = wd^2 = w(144 - w^2) = 144w - w^3 \quad S' = 144 - 3w^2 \quad S'' = -6w$

$144 - 3w^2 = 0 \Rightarrow w = \sqrt{48} \quad S''(\sqrt{48}) = -6\sqrt{48}$

By the 2nd derivative test, the dimensions of the strongest beam that can be cut from a 12-in diameter log are $4\sqrt{3}$ in. by $\sqrt{96} = 4\sqrt{6}$ in.

38)

$w^2 + d^2 = 12^2 \Rightarrow w = \sqrt{144 - d^2}$

$S = wd^3 = \sqrt{144 - d^2}\left(d^3\right)$

$S' = \sqrt{144 - d^2}\left(3d^2\right) + d^3\left(\dfrac{-d}{\sqrt{144 - d^2}}\right) = \dfrac{3d^2\left(144 - d^2\right) - d^4}{\sqrt{144 - d^2}} = \dfrac{432d^2 - 4d^4}{\sqrt{144 - d^2}}$

$\dfrac{432d^2 - 4d^4}{\sqrt{144 - d^2}} = 0 \Rightarrow 432d^2 - 4d^4 = 0 \Rightarrow d = 0, \sqrt{108} = 6\sqrt{3}$

$S'' = \dfrac{\sqrt{144 - d^2}\left(864d - 16d^3\right) + \left(432d^2 - 4d^4\right)\left(\dfrac{-d}{\sqrt{144 - d^2}}\right)}{144 - d^2} \quad S''\left(6\sqrt{3}\right) = -864\sqrt{3}$

By the 2nd derivative test, the maximum stiffness occurs when the 12-in. diameter cylindrical log is cut into a beam that is $6\sqrt{3}$ in. by 6 in.

39) a)

$\dfrac{ds}{dt} = v(t) = -10\pi\sin\pi t \quad \dfrac{dv}{dt} = a(t) = -10\pi^2\cos\pi t$

$-10\pi^2\cos\pi t = 0 \Rightarrow \cos\pi t = 0 \Rightarrow t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$

Maximum speed will be 10π cm/sec

Position: $s\left(\frac{1}{2}\right) = 0$ cm, $s\left(\frac{3}{2}\right) = 0$ cm, $s\left(\frac{5}{2}\right) = 0$ cm, $s\left(\frac{7}{2}\right) = 0$ cm

$a\left(\frac{1}{2}\right) = 0 \text{ cm/sec}^2,\ a\left(\frac{3}{2}\right) = 0 \text{ cm/sec}^2,\ a\left(\frac{5}{2}\right) = 0 \text{ cm/sec}^2,\ a\left(\frac{7}{2}\right) = 0 \text{ cm/sec}^2$

b)

$\dfrac{da}{dt} = 10\pi^3\sin\pi t$

$10\pi^3\sin\pi t = 0 \Rightarrow \sin\pi t = 0 \Rightarrow t = 0, 1, 2, 3, 4$

Position: $s(0) = 10$ cm, $s(1) = -10$ cm, $s(2) = 10$ cm, $s(3) = -10$ cm, $s(4) = 10$ cm

$|v(0)| = 0$ cm/sec, $|v(1)| = 0$ cm/sec, $|v(2)| = 0$ cm/sec, $|v(3)| = 0$ cm/sec, $|v(4)| = 0$ cm/sec

40)

$i' = -2\sin t + 2\cos t \quad i'' = -2\cos t - 2\sin t$

$-2\sin t + 2\cos t = 0 \Rightarrow \cos t = \sin t \Rightarrow t = \frac{\pi}{4} + n\pi$, where n is an integer

$i''\left(\frac{\pi}{4}\right) = -2\cos\left(\frac{\pi}{4}\right) - 2\sin\left(\frac{\pi}{4}\right) = -\sqrt{2} - \sqrt{2} = -2\sqrt{2}$

$i = 2\cos\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$ amps

By the 2nd derivative test, this value is a maximum.

41)

$$D=\sqrt{\left(x-\frac{3}{2}\right)^2+\left(\sqrt{x}-0\right)^2}=\sqrt{x^2-2x+\frac{9}{4}}\Rightarrow D^2=x^2-2x+\frac{9}{4}$$

$$\frac{d}{dx}\left(D^2\right)=2x-2 \quad \frac{d^2}{dx^2}\left(D^2\right)=2$$

$$2x-2=0\Rightarrow x=1$$

$$D(1)=\sqrt{1^2-2(1)+\frac{9}{4}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$$

By the 2nd derivative test, this distance is a minimum.

42)

$$D=\sqrt{\left(x-1\right)^2+\left(\sqrt{16-x^2}-\sqrt{3}\right)^2}=\sqrt{x^2-2x+1+16-x^2-2\sqrt{48-3x^2}-3}=\sqrt{-2x+14-2\sqrt{48-3x^2}}$$

$$\Rightarrow D^2=-2x+14-2\sqrt{48-3x^2}$$

$$\frac{d}{dx}\left(D^2\right)=-2+\frac{6x}{\sqrt{48-3x^2}}=\frac{6x-2\sqrt{48-3x^2}}{\sqrt{48-3x^2}} \quad \frac{d^2}{dx^2}\left(D^2\right)=\frac{288+18x-18x^2}{\left(48-3x^2\right)^{3/2}}$$

$$\frac{6x-2\sqrt{48-3x^2}}{\sqrt{48-3x^2}}=0\Rightarrow 6x-2\sqrt{48-3x^2}=0\Rightarrow 3x=\sqrt{48-3x^2}\Rightarrow 9x^2=48-3x^2\Rightarrow x^2=4\Rightarrow x=2$$

$$D(2)=\sqrt{\left(2-1\right)^2+\left(\sqrt{16-2^2}-\sqrt{3}\right)^2}=\sqrt{1+12-2\sqrt{36}+3}=2$$

$$\frac{d^2}{dx^2}\left(D^2(2)\right)=\frac{288+18(2)-18(2)^2}{\left(48-3(2)^2\right)^{3/2}}=\frac{7}{6}$$

By the 2nd derivative test, the minimum distance is 2.

43)

$$f'(x)=2x-1 \quad f''(x)=2$$

$$2x-1=0\Rightarrow x=\frac{1}{2}$$

$$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2-\frac{1}{2}+1=\frac{3}{4}$$

By the 2nd derivative test, this value is a minimum. Therefore the function cannot be negative.

44)

$$f'(x)=-4\sin x-2\sin 2x \quad f''(x)=-4\cos x-4\cos 2x$$

$$-4\sin x-2\sin 2x=0\Rightarrow -2\sin x=\sin 2x\Rightarrow -2\sin x=2\sin x\cos x\Rightarrow \cos x=-1\Rightarrow x=\pi$$

$$f''(\pi)=-4\cos\pi-4\cos 2\pi=4-4=0$$

$$f'\left(\frac{\pi}{2}\right)=-4\sin\left(\frac{\pi}{2}\right)-2\sin 2\left(\frac{\pi}{2}\right)=-4 \quad f'\left(\frac{3\pi}{2}\right)=-4\sin\left(\frac{3\pi}{2}\right)-2\sin 2\left(\frac{3\pi}{2}\right)=4$$

The 2nd derivative test is inconclusive. The 1st derivative test shows that $x=\pi$ creates a minimum. The minimum value for y is 0, so the function will never be negative.

45) a) When the weights pass each other, $s_1 = s_2$

$$2\sin t = \sin 2t \Rightarrow 2\sin t = 2\sin t\cos t \Rightarrow 2\sin t\cos t - 2\sin t = 0 \Rightarrow 2\sin t(\cos t - 1) = 0$$

$\Rightarrow \sin t = 0, \cos t = 1 \Rightarrow t = n\pi,$ n is an positive integer.

b)

$$d = s_1 - s_2 = 2\sin t - \sin 2t = 2\sin t - 2\sin t\cos t$$

$$d' = 2\cos t - 2\sin t(-\sin t) - 2\cos t(\cos t) = 2\cos t + 2\sin^2 t - 2\cos^2 t$$

$$d'' = -2\sin t + 4\sin t(\cos t) + 4\cos t(\sin t) = -2\sin t + 8\sin t\cos t$$

$$2\cos t + 2\sin^2 t - 2\cos^2 t = 0 \Rightarrow 2\cos t + 2(1 - \cos^2 t) - 2\cos^2 t = 0$$

$$\Rightarrow 2\cos t + 2 - 4\cos^2 t = 0 \Rightarrow -2(2\cos^2 t - \cos t - 1) = 0 \Rightarrow 2(2\cos t + 1)(\cos t - 1) = 0$$

$$\Rightarrow \cos t = -\frac{1}{2} \text{ or } \cos t = 1 \Rightarrow t = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ or } 0$$

$$d''\left(\frac{2\pi}{3}\right) = -2\sin\left(\frac{2\pi}{3}\right) + 8\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right) = -2\left(\frac{\sqrt{3}}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = -3\sqrt{3}$$

$$d''\left(\frac{4\pi}{3}\right) = -2\sin\left(\frac{4\pi}{3}\right) + 8\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{4\pi}{3}\right) = -2\left(\frac{\sqrt{3}}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = -3\sqrt{3}$$

$$d''(0) = -2\sin(0) + 8\sin(0)\cos(0) = 0$$

$$d\left(\frac{2\pi}{3}\right) = 2\sin\left(\frac{2\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = \frac{3\sqrt{3}}{2}$$

$$d\left(\frac{5\pi}{3}\right) = 2\sin\left(\frac{4\pi}{3}\right) - 2\sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{4\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = \frac{3\sqrt{3}}{2}$$

By the 2nd derivative test, the distance is greatest when $t = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. The greatest distance is $\frac{3\sqrt{3}}{2}$

46) a)

$$\sin t = \sin\left(t + \frac{\pi}{3}\right) \Rightarrow \sin t = \sin t\cos\left(\frac{\pi}{3}\right) + \cos t\sin\left(\frac{\pi}{3}\right) \Rightarrow \sin t = \frac{\sin t}{2} + \frac{\sqrt{3}}{2}\cos t$$

$$\Rightarrow \sin t = \sqrt{3}\cos t \Rightarrow \tan t = \sqrt{3} \Rightarrow t = \frac{\pi}{3}, \frac{4\pi}{3}$$

46) b)

$$d = \sin t - \sin\left(t + \frac{\pi}{3}\right) \Rightarrow d = \sin t - \sin t \cos\left(\frac{\pi}{3}\right) - \cos t \sin\left(\frac{\pi}{3}\right) \Rightarrow d = \sin t - \frac{\sin t}{2} - \frac{\sqrt{3}}{2}\cos t$$

$$\Rightarrow d = \frac{\sin t}{2} - \frac{\sqrt{3}}{2}\cos t$$

$$d' = \frac{\cos t}{2} + \frac{\sqrt{3}}{2}\sin t \quad d'' = -\frac{\sin t}{2} + \frac{\sqrt{3}}{2}\cos t$$

$$\frac{\cos t}{2} + \frac{\sqrt{3}}{2}\sin t = 0 \Rightarrow \sqrt{3}\sin t = -\cos t \Rightarrow \tan t = -\frac{1}{\sqrt{3}} \Rightarrow t = \frac{5\pi}{6}, \frac{11\pi}{6}$$

$$d''\left(\frac{5\pi}{6}\right) = -\frac{\sin\left(\frac{5\pi}{6}\right)}{2} + \frac{\sqrt{3}}{2}\cos\left(\frac{5\pi}{6}\right) = -\frac{1}{4} + \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right) = -1$$

$$d''\left(\frac{11\pi}{6}\right) = -\frac{\sin\left(\frac{11\pi}{6}\right)}{2} + \frac{\sqrt{3}}{2}\cos\left(\frac{11\pi}{6}\right) = \frac{1}{4} + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right) = 1$$

$$d = \sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{5\pi}{6} + \frac{\pi}{3}\right) = \frac{1}{2} - \sin\left(\frac{7\pi}{6}\right) = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

By the 2nd derivative test, this is a maximum value.

46) c)

$$d'' = -\frac{\sin t}{2} + \frac{\sqrt{3}}{2}\cos t$$

$$-\frac{\sin t}{2} + \frac{\sqrt{3}}{2}\cos t = 0 \Rightarrow \sqrt{3}\cos t = \sin t \Rightarrow \tan t = \sqrt{3} \Rightarrow t = \frac{\pi}{3}, \frac{4\pi}{3}$$

47)