Calculus page 470 ff Review Solutions

7) $\lim_{n\to\infty}\frac{3n^2-1}{2n^2+1}=\lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{2+\frac{1}{n^2}}=\frac{3}{2}$, so the sequence converges.

8) $\lim_{n\to\infty}(-1)^n\frac{3n-1}{n+2}=\lim_{n\to\infty}(-1)^n\frac{3-\frac{1}{n}}{1+\frac{2}{n}}=\pm 3$, so the sequence diverges.

9) $\lim_{t\to 0}\frac{t-\ln(1+2t)}{t^2}\quad\frac{0}{0}\quad(\text{L'H})\lim_{t\to 0}\frac{1-\frac{2}{1+2t}}{2t}=\infty$, so the limit does not exist.

10) $\lim_{t\to 0}\frac{\tan 3t}{\tan 5t}\quad\frac{0}{0}\quad(\text{L'H})\lim_{t\to 0}\frac{3\sec^2(3t)}{5\sec^2(5t)}=\frac{3}{5}$

11) $\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\quad\frac{0}{0}\quad(\text{L'H})\lim_{x\to 0}\frac{x\cos x+\sin x}{\sin x}\quad\frac{0}{0}\quad(\text{L'H})\lim_{x\to 0}\frac{-x\sin x+\cos x+\cos x}{\cos x}=2$

12) $\lim_{x\to 1}x^{1/(1-x)}\quad 1^\infty\quad y=x^{1/(1-x)}\quad \ln y=\frac{1}{1-x}\ln x\quad \lim_{x\to 1}\frac{\ln x}{1-x}\quad\frac{0}{0}\quad(\text{L'H})\lim_{x\to 1}\frac{\frac{1}{x}}{-1}=-1$

$\lim_{x\to 1}x^{1/(1-x)}=e^{-1}$

13) $\lim_{x\to\infty}x^{1/x}\quad\infty^0\quad y=x^{1/x}\quad \ln y=\frac{1}{x}\ln x\quad \lim_{x\to\infty}\frac{\ln x}{x}\quad\frac{\infty}{\infty}\quad(\text{L'H})\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$

$\lim_{x\to\infty}x^{1/x}=e^0=1$

14) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x\quad 1^\infty\quad y=\left(1+\frac{3}{x}\right)^x\quad \ln y=x\ln\left(1+\frac{3}{x}\right)\quad \lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}\quad\frac{0}{0}\quad(\text{L'H})\lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}}=3$

$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x=e^3$

15) $\lim_{r\to\infty}\frac{\cos r}{\ln r}=0$

16) $\lim_{\theta\to\pi/2}\left(\theta-\frac{\pi}{2}\right)\sec\theta \quad (\text{RW})\lim_{\theta\to\pi/2}\frac{\theta-\frac{\pi}{2}}{\cos\theta} \quad \frac{0}{0} \qquad (\text{L'H})\lim_{\theta\to\pi/2}\frac{1}{-\sin\theta}=-1$

17) $\lim_{x\to1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right) \quad (\text{RW})\lim_{x\to1}\frac{\ln x-x+1}{(x-1)\ln x} \quad \frac{0}{0} \qquad (\text{L'H})\lim_{x\to1}\frac{\frac{1}{x}-1}{\frac{x-1}{x}+\ln x}=(\text{RW})\lim_{x\to1}\frac{1-x}{x-1+x\ln x}$

$\frac{0}{0} \qquad (\text{L'H})\lim_{x\to1}\frac{-1}{1+1+\ln x}=-\frac{1}{2}$

18) $\lim_{x\to0^+}\left(1+\frac{1}{x}\right)^x \quad \infty^0 \qquad y=\left(1+\frac{1}{x}\right)^x \quad \ln y=x\left(\ln\left(1+\frac{1}{x}\right)\right) \quad \lim_{x\to0^+}x\left(\ln\left(1+\frac{1}{x}\right)\right) \quad 0\cdot\infty$

$(\text{RW})\lim_{x\to0^+}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \frac{\infty}{\infty} \quad (\text{L'H})\lim_{x\to0^+}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}=(\text{RW})\lim_{x\to0^+}\frac{1}{1+\frac{1}{x}}=0$

$\lim_{x\to0^+}y=e^0=1$

19) $\lim_{\theta\to0^+}(\tan\theta)^\theta \quad 0^0 \qquad y=(\tan\theta)^\theta \quad \ln y=\theta\ln(\tan\theta) \quad \lim_{x\to0^+}\theta\ln(\tan\theta) \quad 0\cdot\infty$

$(\text{RW})\lim_{x\to0^+}\frac{\ln(\tan\theta)}{\frac{1}{\theta}} \quad \frac{\infty}{\infty} \quad (\text{L'H})\lim_{x\to0^+}\frac{\frac{1}{\tan\theta}(\sec^2\theta)}{\left(-\frac{1}{\theta^2}\right)}=(\text{RW})\lim_{x\to0^+}\frac{-\theta^2}{\sin\theta\cos\theta} \quad \frac{0}{0}$

$(\text{L'H})\lim_{x\to0^+}\frac{-2\theta}{\sin\theta(-\sin\theta)+\cos\theta\cos\theta}=0 \quad \lim_{x\to0^+}y=e^0=1$

20) $\lim_{\theta\to\infty}\theta^2\sin\left(\frac{1}{\theta}\right) \quad \infty\cdot0 \qquad (\text{RW})\lim_{\theta\to\infty}\frac{\sin\left(\frac{1}{\theta}\right)}{\frac{1}{\theta^2}} \quad \frac{0}{0} \quad (\text{L'H})\lim_{\theta\to\infty}\frac{\left(\frac{-1}{\theta^2}\right)\left(\cos\left(\frac{1}{\theta}\right)\right)}{\left(-\frac{2}{\theta^3}\right)}=(\text{RW})\lim_{\theta\to\infty}\left(\frac{\theta^2}{2}\cos\left(\frac{1}{\theta}\right)\right)=\infty$

21) $\lim_{x\to\infty}\frac{x^3-3x^2+1}{2x^2+x-3} \quad \frac{\infty}{\infty} \qquad (\text{L'H})\lim_{x\to\infty}\frac{3x^2-6x}{4x+1} \quad \frac{\infty}{\infty} \quad (\text{L'H})\lim_{x\to\infty}\frac{6x-6}{4}=\infty$

22) $\lim_{x\to\infty}\frac{3x^2-x+1}{x^4-x^3+2}\quad\frac{\infty}{\infty}\qquad(\text{L'H})\lim_{x\to\infty}\frac{6x+1}{4x^3-3x^2}\quad\frac{\infty}{\infty}\quad(\text{L'H})\lim_{x\to\infty}\frac{6}{12x^2-6x}=0$

23) $\lim_{x\to\infty}\frac{x}{5x}=\frac{1}{5}$. As this is a nonzero constant, they grow at the same rate.

24) $\lim_{x\to\infty}\frac{\log_2 x}{\log_3 x}=(\text{L'H})\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}}=\frac{\ln 3}{\ln 2}$. As this is a nonzero constant, they grow at the same rate.

25) $\lim_{x\to\infty}\frac{x}{x+\frac{1}{x}}=(\text{RW})\lim_{x\to\infty}\frac{x}{\frac{x^2+1}{x}}=(\text{RW})\lim_{x\to\infty}\frac{x^2}{x^2+1}=(\text{L'H})\lim_{x\to\infty}\frac{2x}{2x}=1$

As this is a nonzero constant, they grow at the same rate.

26) $\lim_{x\to\infty}\frac{\frac{x}{100}}{xe^{-x}}=(\text{RW})\lim_{x\to\infty}\frac{xe^x}{100x}=(\text{L'H})\lim_{x\to\infty}\frac{xe^x+e^x}{100}=\infty$

$\frac{x}{100}$ grows faster than xe^{-x}

27) $\lim_{x\to\infty}\frac{x}{\tan^{-1}x}=\infty$

x grows faster than $\tan^{-1}x$

28)

$$\lim_{x\to\infty}\frac{\csc^{-1}x}{\frac{1}{x}}=(\text{RW})\lim_{x\to\infty}\frac{\sin^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}}=(\text{L'H})\lim_{x\to\infty}\frac{\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=1$$

$\csc^{-1}x$ grows at the same rate as $\frac{1}{x}$

29) $\lim_{x\to\infty}\frac{x^{\ln x}}{x^{\log_2 x}}\quad\frac{\infty}{\infty}\quad(\text{RW})\lim_{x\to\infty}\frac{x^{\ln x}}{x^{\frac{\ln x}{\ln 2}}}=(\text{RW})\lim_{x\to\infty}\left(\frac{x}{x^{\ln 2}}\right)^{\ln x}=0$

$x^{\ln x}$ grows slower than $x^{\log_2 x}$

30) $\lim_{x\to\infty}\frac{3^{-x}}{2^{-x}}\quad\frac{0}{0}\quad(\text{RW})\lim_{x\to\infty}\frac{2^x}{3^x}=(\text{RW})\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0$

3^{-x} grows slower than 2^{-x}

31) $\lim_{x\to\infty}\frac{\ln 2x}{\ln x^2} \quad \frac{\infty}{\infty} \quad (\text{L'H})\lim_{x\to\infty}\frac{\frac{2}{2x}}{\frac{2x}{x^2}} = (\text{RW})\lim_{x\to\infty}\frac{2x^2}{4x^2} = \frac{1}{2}$

$\ln 2x$ grows at the same rate as $\ln x^2$

32) $\lim_{x\to\infty}\frac{10x^3+2x}{e^x} \quad \frac{\infty}{\infty} \quad (\text{L'H})\lim_{x\to\infty}\frac{30x^2+2}{e^x} \quad \frac{\infty}{\infty} \quad (\text{L'H})\lim_{x\to\infty}\frac{60x}{e^x} \quad \frac{\infty}{\infty} \quad (\text{L'H})\lim_{x\to\infty}\frac{60}{e^x} = 0$

$10x^3+2x$ grows slower than e^x

33) $\lim_{x\to\infty}\frac{\tan^{-1}\frac{1}{x}}{\frac{1}{x}} \quad \frac{0}{0} \quad (\text{L'H})\lim_{x\to\infty}\frac{\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = (\text{RW})\lim_{x\to\infty}\frac{1}{1+\left(\frac{1}{x}\right)^2} = 1$

$\tan^{-1}\frac{1}{x}$ grows at the same rate as $\frac{1}{x}$

34) $\lim_{x\to\infty}\frac{\sin^{-1}\frac{1}{x}}{\frac{1}{x^2}} \quad \frac{0}{0} \quad (\text{L'H})\lim_{x\to\infty}\frac{\frac{1}{\sqrt{1-\left(\frac{1}{x^2}\right)}}\left(-\frac{1}{x^2}\right)}{\left(-\frac{2}{x^3}\right)} = (\text{RW})\lim_{x\to\infty}\frac{x}{2\sqrt{1-\left(\frac{1}{x^2}\right)}} = \infty$

$\sin^{-1}\frac{1}{x}$ grows faster than $\frac{1}{x^2}$

35a) $\lim_{x\to 0}\frac{2^{\sin x}-1}{e^x-1} \quad \frac{0}{0} \quad (\text{L'H})\lim_{x\to 0}\frac{2^{\sin x}(\ln 2)(\cos x)}{e^x} = \ln 2$

35b) $f(x) = \begin{cases} \frac{2^{\sin x}-1}{e^x-1}, & x \neq 0 \\ \ln 2 & x = 0 \end{cases}$

36a) $\lim_{x\to 0^+} x\ln x \quad (\text{RW})\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} \quad \frac{0}{0} \quad (\text{L'H})\lim_{x\to 0^+}\frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = (\text{RW})\lim_{x\to 0^+}(-x) = 0$

36b) $f(x) = \begin{cases} x\ln x, & x \neq 0 \\ 0 & x = 0 \end{cases}$

37) $$\int_1^{\infty}\frac{dx}{x^{3/2}}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x^{3/2}}=\lim_{b\to\infty}\left[-\frac{2}{\sqrt{x}}\right]_1^b=\lim_{b\to\infty}\left[-\frac{2}{\sqrt{b}}-(-2)\right]=2$$

38) $$\int_1^{\infty}\frac{dx}{x^2+7x+12}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x^2+7x+12}\qquad \frac{1}{x^2+7x+12}=\frac{A}{x+3}+\frac{B}{x+4}=\frac{1}{x+3}-\frac{1}{x+4}$$

$$\lim_{b\to\infty}\int_1^{b}\frac{dx}{x^2+7x+12}=\lim_{b\to\infty}\int_1^{b}\left(\frac{1}{x+3}-\frac{1}{x+4}\right)dx=\lim_{b\to\infty}\Big[\ln|x+3|-\ln|x+4|\Big]_1^b=\lim_{b\to\infty}\left[\ln\left|\frac{x+3}{x+4}\right|\right]_1^b$$

$$=\lim_{b\to\infty}\left[\ln\left|\frac{b+3}{b+4}\right|-\ln\left(\frac{4}{5}\right)\right]=(\mathrm{RW})\lim_{b\to\infty}\left[\ln\left|\frac{1+\frac{3}{b}}{1+\frac{4}{b}}\right|-\ln\left(\frac{4}{5}\right)\right]=-\ln\left(\frac{4}{5}\right)$$

39) $$\int_{-\infty}^{-1}\frac{3dx}{3x-x^2}=\lim_{b\to-\infty}\int_b^{-1}\frac{3dx}{3x-x^2}\qquad \frac{3}{3x-x^2}=\frac{A}{x}+\frac{B}{3-x}=\frac{1}{x}+\frac{1}{3-x}$$

$$\lim_{b\to-\infty}\int_b^{-1}\frac{3dx}{3x-x^2}=\lim_{b\to-\infty}\int_b^{-1}\left(\frac{1}{x}+\frac{1}{3-x}\right)dx=\lim_{b\to-\infty}\Big[\ln|x|-\ln|3-x|\Big]_b^{-1}=\lim_{b\to-\infty}\left[\ln\left|\frac{x}{3-x}\right|\right]_b^{-1}$$

$$=\lim_{b\to-\infty}\left[\ln\left(\frac{1}{4}\right)-\ln\left|\frac{-b}{3+b}\right|\right]=(\mathrm{RW})\lim_{b\to-\infty}\left[\ln\left(\frac{1}{4}\right)-\ln\left|\frac{-1}{\frac{3}{b}+1}\right|\right]=\ln\left(\frac{1}{4}\right)$$

40) $$\int_0^{3}\frac{dx}{\sqrt{9-x^2}}=\lim_{b\to3}\int_0^{b}\frac{dx}{\sqrt{9-x^2}}=(\mathrm{RW})\lim_{b\to3}\int_0^{b}\frac{dx}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}=(\mathrm{RW})\lim_{b\to3}\int_0^{b}\frac{dx}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}$$

$$=\lim_{b\to3}\left[\sin^{-1}\frac{x}{3}\right]_0^b=\lim_{b\to3}\left(\sin^{-1}\left(\frac{b}{3}\right)-\sin^{-1}0\right)=\frac{\pi}{2}$$

41) $$\int_0^{1}\ln(x)\,dx=\lim_{b\to0^+}\int_b^{1}\ln(x)\,dx\qquad \begin{matrix}u=\ln x & dv=dx\\ du=\frac{1}{x}dx & v=x\end{matrix}\qquad \int\ln(x)\,dx=x\ln x-x+C$$

$$\lim_{b\to0^+}\int_b^{1}\ln(x)\,dx=\lim_{b\to0^+}\Big[x\ln x-x\Big]_b^1=\lim_{b\to0^+}\Big[0-1-(b\ln b-b)\Big]_0^b$$

$$=-1-\lim_{b\to0^+}\left(\frac{\ln b}{\frac{1}{b}}\right)=-1-(\mathrm{L'H})\lim_{b\to0^+}\left(\frac{\frac{1}{b}}{-\frac{1}{b^2}}\right)=-1$$

42) $$\int_{-1}^{1}\frac{dy}{y^{2/3}}=\lim_{a\to0}\int_{-1}^{a}\frac{dy}{y^{2/3}}+\lim_{b\to0}\int_b^{1}\frac{dy}{y^{2/3}}=\lim_{a\to0}\Big[3y^{1/3}\Big]_{-1}^{a}+\lim_{b\to0}\Big[3y^{1/3}\Big]_b^1$$

$$=\lim_{a\to0}\Big[3a^{1/3}-(-3)\Big]+\lim_{b\to0}\Big[3-3b^{1/3}\Big]=3+3=6$$

43) $$\int_{-2}^{0}\frac{d\theta}{(\theta+1)^{3/5}}=\lim_{a\to-1}\int_{-2}^{a}\frac{d\theta}{(\theta+1)^{3/5}}+\lim_{b\to-1}\int_{b}^{0}\frac{d\theta}{(\theta+1)^{3/5}}=\lim_{a\to-1}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_{-2}^{a}+\lim_{b\to-1}\left[\frac{5}{2}(\theta+1)^{2/5}\right]_{b}^{0}$$

$$=\lim_{a\to-1}\left[\frac{5}{2}(a+1)^{2/5}-\frac{5}{2}\right]+\lim_{b\to-1}\left[\frac{5}{2}-\frac{5}{2}(b+1)^{2/5}\right]=-\frac{5}{2}+\frac{5}{2}=0$$

44) $$\int_{3}^{\infty}\frac{2dx}{x^2-2x}=\lim_{b\to\infty}\int_{3}^{b}\frac{2dx}{x^2-2x}\qquad\frac{2}{x^2-2x}=\frac{A}{x}+\frac{B}{x-2}=-\frac{1}{x}+\frac{1}{x-2}$$

$$\lim_{b\to\infty}\int_{3}^{b}\frac{2dx}{x^2-2x}=\lim_{b\to\infty}\int_{3}^{b}\left(-\frac{1}{x}+\frac{1}{x-2}\right)dx=\lim_{b\to\infty}\left[-\ln|x|+\ln|x-2|\right]_{3}^{b}=\lim_{b\to\infty}\left[\ln\left|\frac{x-2}{x}\right|\right]_{3}^{b}$$

$$=\lim_{b\to\infty}\left[\ln\left|\frac{b-2}{b}\right|-\ln\left(\frac{1}{3}\right)\right]=(\mathrm{RW})\lim_{b\to-\infty}\left[\ln\left|\frac{1-\frac{2}{b}}{1}\right|-\ln\frac{1}{3}\right]=0-\ln\left(\frac{1}{3}\right)=-\ln\frac{1}{3}=\ln 3$$

45) $$\int_{0}^{\infty}x^2e^{-x}dx=\lim_{a\to\infty}\int_{0}^{a}x^2e^{-x}dx\qquad\int x^2e^{-x}dx=-x^2e^{-x}-2xe^{-x}-2e^{-x}+c$$

$$=\lim_{a\to\infty}\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]_{0}^{a}=\lim_{a\to\infty}\left[-a^2e^{-a}-2ae^{-a}-2e^{-a}-(0-0-2)\right]=2$$

46) $$\int_{-\infty}^{0}xe^{3x}\,dx=\lim_{b\to-\infty}\int_{b}^{0}xe^{3x}\,dx\qquad\begin{matrix}u=x & dv=e^{3x}dx\\ du=dx & v=\frac{1}{3}e^{3x}\end{matrix}\qquad\int xe^{3x}\,dx=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+C$$

$$\lim_{b\to-\infty}\int_{b}^{0}xe^{3x}\,dx=\lim_{b\to-\infty}\left[\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}\right]_{b}^{0}=\lim_{b\to-\infty}\left[0-\frac{1}{9}-\left(\frac{1}{3}be^{3b}-\frac{1}{9}e^{3b}\right)\right]=-\frac{1}{9}$$

47) $$\int_{-\infty}^{\infty}\frac{dx}{e^x+e^{-x}}=\lim_{b\to-\infty}\int_{b}^{0}\frac{dx}{e^x+e^{-x}}+\lim_{a\to\infty}\int_{0}^{a}\frac{dx}{e^x+e^{-x}}\qquad\int\frac{dx}{e^x+e^{-x}}=(\mathrm{RW})\int\frac{dx}{\frac{e^{2x}+1}{e^x}}=(\mathrm{RW})\int\frac{e^xdx}{e^{2x}+1}$$

$$\lim_{b\to-\infty}\int_{b}^{0}\frac{e^xdx}{e^{2x}+1}+\lim_{a\to\infty}\int_{0}^{a}\frac{e^xdx}{e^{2x}+1}\qquad\begin{matrix}u=e^x\\ du=e^xdx\end{matrix}$$

$$=\lim_{b\to-\infty}\int_{e^b}^{1}\frac{du}{u^2+1}+\lim_{a\to\infty}\int_{1}^{e^a}\frac{du}{u^2+1}=\lim_{b\to-\infty}\left[\tan^{-1}u\right]_{e^b}^{1}+\lim_{a\to\infty}\left[\tan^{-1}u\right]_{1}^{e^a}$$

$$=\lim_{b\to-\infty}\left[\tan^{-1}1-\tan^{-1}e^b\right]+\lim_{a\to\infty}\left[\tan^{-1}e^a-\tan^{-1}1\right]=\left(\frac{\pi}{4}-\left(-\frac{\pi}{2}\right)\right)+\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{2}$$

48) $$\int_{-\infty}^{\infty}\frac{4dx}{x^2+16}=(\mathrm{RW})\int_{-\infty}^{\infty}\frac{4dx}{\left(\frac{x}{4}\right)^2+1}\qquad\begin{matrix}u=\frac{x}{4}\\ du=\frac{1}{4}dx\end{matrix}$$

$$=\lim_{b\to-\infty}\int_{b/4}^{0}\frac{du}{u^2+1}+\lim_{a\to\infty}\int_{0}^{a/4}\frac{du}{u^2+1}=\lim_{b\to-\infty}\left[\tan^{-1}u\right]_{b/4}^{0}+\lim_{a\to\infty}\left[\tan^{-1}u\right]_{0}^{a/4}$$

$$=\lim_{b\to-\infty}\left[0-\tan^{-1}\frac{b}{4}\right]+\lim_{a\to\infty}\left[\tan^{-1}\frac{a}{4}-0\right]=-\left(\left(-\frac{\pi}{2}\right)\right)+\left(\frac{\pi}{2}\right)=\pi$$