Calculus Section 11.3 Solutions

1) a) $(1,1)$ b) $(1,0)$ c) $(0,0)$ d) $(-1,-1)$

2) a) $\left(\frac{3\sqrt{3}}{2},-\frac{3}{2}\right)$ b) $(3,4)$ c) $(1,0)$ d) $\left(-\sqrt{3},3\right)$

3) a) $\left(\sqrt{2},\frac{3\pi}{4}\right)$ or $\left(\sqrt{2},-\frac{5\pi}{4}\right)$ or $\left(-\sqrt{2},-\frac{\pi}{4}\right)$ or $\left(-\sqrt{2},\frac{7\pi}{4}\right)$

b) $\left(2,-\frac{\pi}{3}\right)$ or $\left(-2,\frac{2\pi}{3}\right)$ or $\left(2,\frac{5\pi}{3}\right)$ or $\left(-2,-\frac{4\pi}{3}\right)$

c) $\left(3,\frac{\pi}{2}\right)$ or $\left(3,-\frac{3\pi}{2}\right)$ or $\left(-3,-\frac{\pi}{2}\right)$ or $\left(-3,\frac{3\pi}{2}\right)$

d) $(1,\pi)$ or $(1,-\pi)$ or $(-1,0)$ or $(-1,2\pi)$

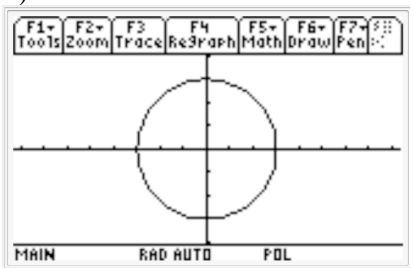
4) a) $\left(2,\frac{7\pi}{6}\right)$ or $\left(2,-\frac{5\pi}{6}\right)$ or $\left(-2,\frac{\pi}{6}\right)$ or $\left(-2,-\frac{11\pi}{6}\right)$

b) $\left(5,\tan^{-1}\frac{4}{3}\right)$ or $\left(-5,\pi+\tan^{-1}\frac{4}{3}\right)$

c) $\left(2,\frac{3\pi}{2}\right)$ or $\left(2,-\frac{\pi}{2}\right)$ or $\left(-2,\frac{\pi}{2}\right)$ or $\left(-2,-\frac{3\pi}{2}\right)$

d) $(2,0)$ or $(2,2\pi)$ or $(-2,\pi)$ or $(-2,-\pi)$

5)



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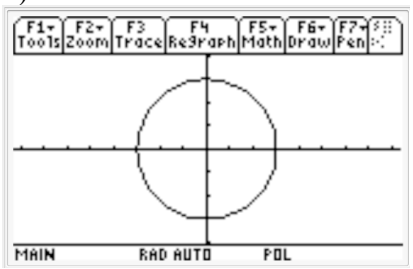
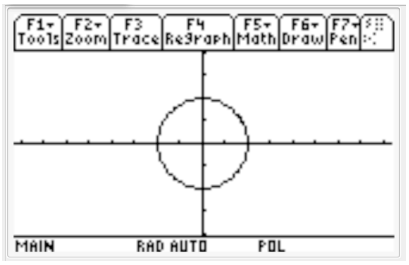


7)

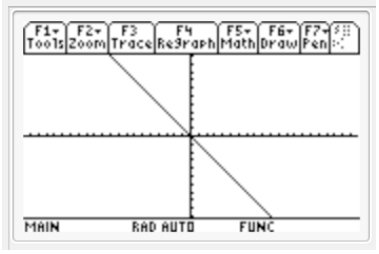
$x^2+y^2=4$



8)

$\tan\theta=\frac{y}{x}\Rightarrow y=x\tan\theta$

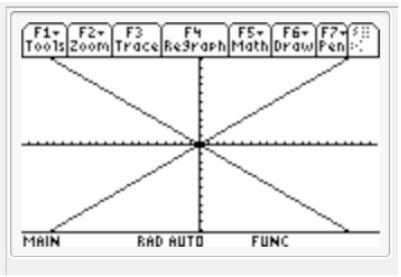
$y=x\tan\left(-\frac{\pi}{4}\right)\Rightarrow y=-x$



9)

$$\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$$

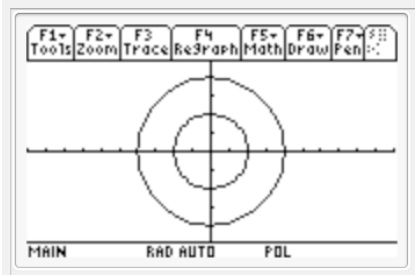
$$y = x\tan\left(\pm\frac{\pi}{6}\right) \Rightarrow y = \pm\frac{\sqrt{3}}{3}x$$



10)

$$r^2 + 8 = 6r \Rightarrow r^2 - 6r + 8 = 0 \Rightarrow$$

$$(r-4)(r-2) = 0 \Rightarrow r = 2 \text{ or } r = 4$$



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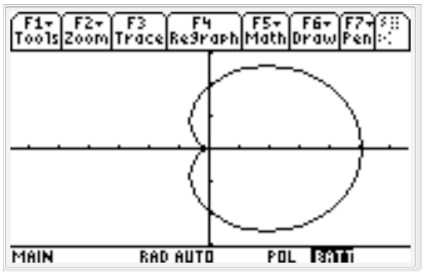

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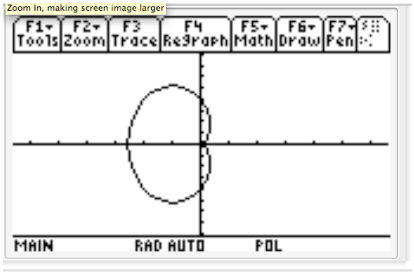

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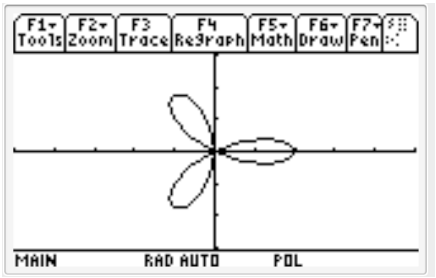

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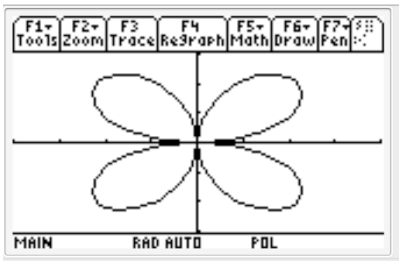

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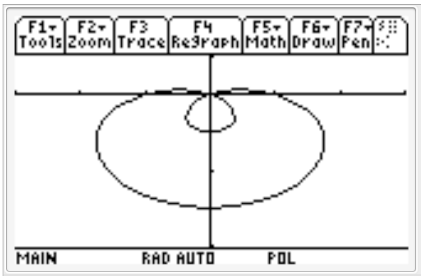

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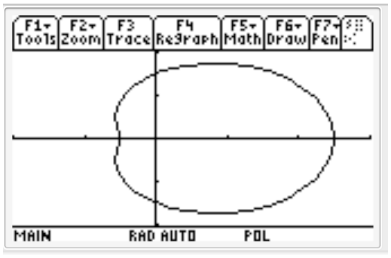

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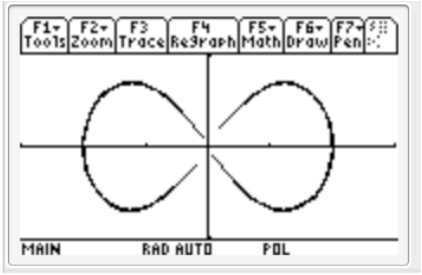

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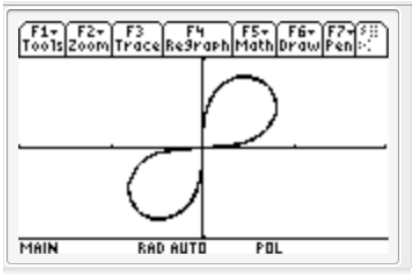
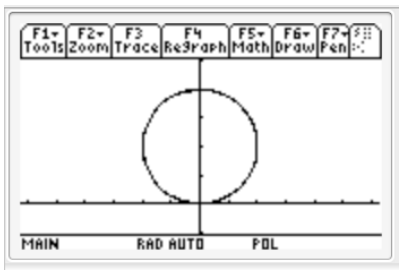
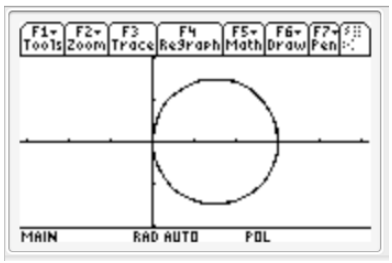


19) 



21) $r = \dfrac{4}{\sin\theta} \Rightarrow r\sin\theta = 4 \Rightarrow y = 4$. This is a horizontal line.

22) $r = \dfrac{-3}{\cos\theta} \Rightarrow r\cos\theta = -3 \Rightarrow x = -3$. This is a vertical line.

23) $x + y = 1$. This is a line with a slope of 1 and a y-intercept of 1.

24) $x^2 + y^2 = 1$. This is a circle with center (0,0) and a radius of 1.

25) $r\sin\theta - 2r\cos\theta = 5 \Rightarrow y - 2x = 5$. This is a line with a slope of 2 and a y-intercept of 5.

26) $r^2\left(2\sin\theta\cos\theta\right) = 2 \Rightarrow r^2(2)\left(\dfrac{y}{r}\right)\left(\dfrac{x}{r}\right) = 2 \Rightarrow xy = 1 \Rightarrow y = \dfrac{1}{x}$. This is a hyperbola.

27) $\left(\dfrac{x}{r}\right)^2 = \left(\dfrac{y}{r}\right)^2 \Rightarrow x^2 = y^2$. This is two lines: $y = x$ and $y = -x$.

28) $x^2 + y^2 \Rightarrow -4x \Rightarrow \left(x^2 + 4x + 4\right) + y^2 = 4 \Rightarrow \left(x + 2\right)^2 + y^2 = 4$. This is a circle, center (-2,0), radius 2.

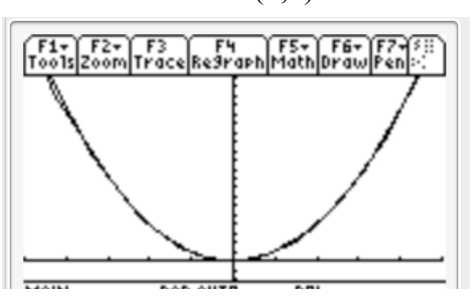
29) $r = 8\left(\dfrac{y}{r}\right) \Rightarrow r^2 = 8y \Rightarrow x^2 + y^2 = 8y \Rightarrow x^2 + \left(y^2 - 8y + 16\right) = 16 \Rightarrow x^2 + \left(y - 4\right)^2 = 16.$

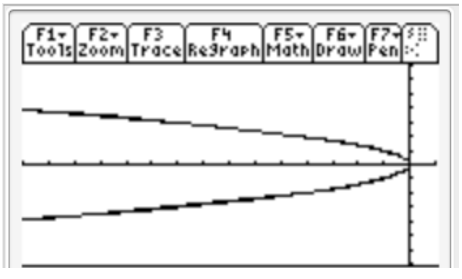
This is a circle with center (0,4) and radius 4

30) $r = 2\left(\dfrac{x}{r}\right) + 2\left(\dfrac{y}{r}\right) \Rightarrow r^2 = 2x + 2y \Rightarrow x^2 + y^2 = 2x + 2y$

$\Rightarrow \left(x^2 - 2x + 1\right) + \left(y^2 - 2y + 1\right) = 2 \Rightarrow \left(x - 1\right)^2 + \left(y - 1\right)^2 = 2$

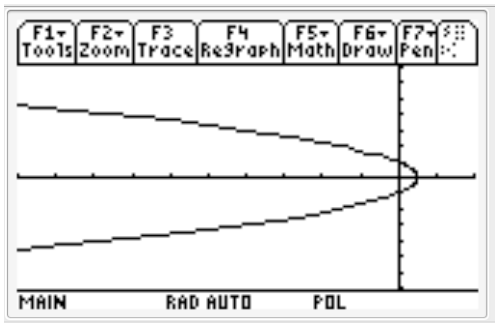
This is a circle with center (1,1) and radius √2.

31) 

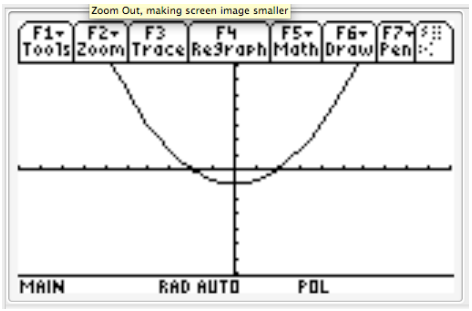
32) 

Parabola

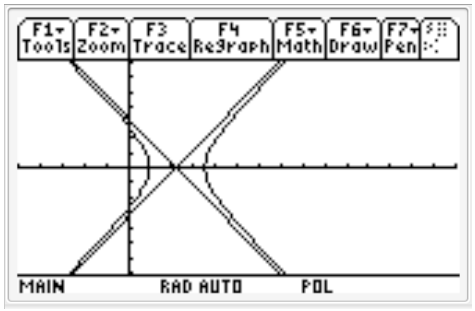
Parabola

33) 

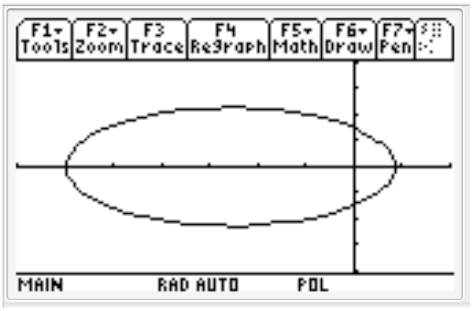
Parabola

34) 

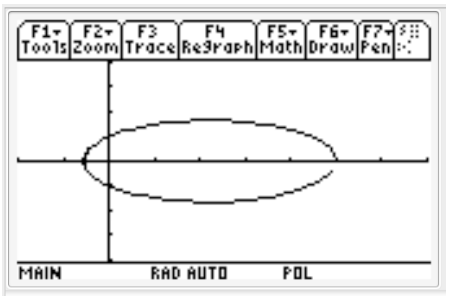
Parabola

35) 

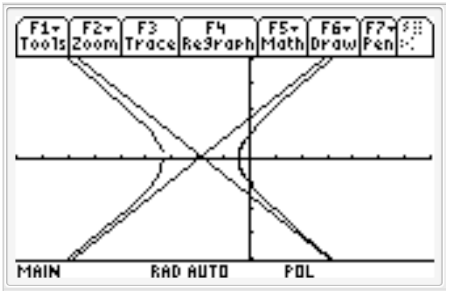
Hyperbola

36) 

Ellipse

37) 

Ellipse

38) 

Hyperbola

39) $y = \left(-1+\sin\theta\right)\sin\theta \Rightarrow \dfrac{dy}{d\theta} = -\cos\theta + 2\sin\theta\cos\theta$

$x = \left(-1+\sin\theta\right)\cos\theta \Rightarrow \dfrac{dx}{d\theta} = \sin\theta + \cos^2\theta - \sin^2\theta$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta\left(2\sin\theta - 1\right)}{\sin\theta + \cos^2\theta - \sin^2\theta}$

$\theta = 0: \quad \dfrac{dy}{dx} = \dfrac{1(0-1)}{0+1-0} = -1$

$\theta = \pi: \quad \dfrac{dy}{dx} = \dfrac{-1(0-1)}{0+1-0} = 1$

40) $y=\left(\cos 2\theta\right)\sin\theta \Rightarrow \dfrac{dy}{d\theta}=\cos 2\theta\cos\theta-2\sin 2\theta\sin\theta$

$x=\left(\cos 2\theta\right)\cos\theta \Rightarrow \dfrac{dx}{d\theta}=-\cos 2\theta\sin\theta-2\sin 2\theta\cos\theta$

$\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{\cos 2\theta\cos\theta-2\sin 2\theta\sin\theta}{-\cos 2\theta\sin\theta-2\sin 2\theta\cos\theta}$

$\theta=0:\quad \dfrac{dy}{dx}=\dfrac{1-0}{0-0}=$ undefined

$\theta=\pi:\quad \dfrac{dy}{dx}=\dfrac{1-0}{0-0}=$ undefined

$\theta=\dfrac{\pi}{2}:\quad \dfrac{dy}{dx}=\dfrac{0-0}{1-0}=0$

$\theta=-\dfrac{\pi}{2}:\quad \dfrac{dy}{dx}=\dfrac{0-0}{1-0}=0$

41) $y=\left(2-3\sin\theta\right)\sin\theta \Rightarrow \dfrac{dy}{d\theta}=2\cos\theta-6\sin\theta\cos\theta$

$x=\left(2-3\sin\theta\right)\cos\theta \Rightarrow \dfrac{dx}{d\theta}=-2\sin\theta-3\left(\cos^2\theta-\sin^2\theta\right)$

$\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{2\cos\theta-6\sin\theta\cos\theta}{-2\sin\theta-3\left(\cos^2\theta-\sin^2\theta\right)}$

$\theta=0:\quad \dfrac{dy}{dx}=\dfrac{2-0}{0-3}=-\dfrac{2}{3}$

$\theta=\pi:\quad \dfrac{dy}{dx}=\dfrac{-2-0}{0-3}=\dfrac{2}{3}$

$\theta=\dfrac{\pi}{2}:\quad \dfrac{dy}{dx}=\dfrac{0-0}{-2+3}=0$

$\theta=\dfrac{3\pi}{2}:\quad \dfrac{dy}{dx}=\dfrac{0-0}{2+3}=0$

42) $y=\left(3-3\cos\theta\right)\sin\theta \Rightarrow \dfrac{dy}{d\theta}=3\cos\theta-3\left(\cos^2\theta-\sin^2\theta\right)$

$x=\left(3-3\cos\theta\right)\cos\theta \Rightarrow \dfrac{dx}{d\theta}=-3\sin\theta+6\cos\theta\sin\theta$

$\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{3\cos\theta-3\left(\cos^2\theta-\sin^2\theta\right)}{-3\sin\theta+6\cos\theta\sin\theta}$

$\theta=\dfrac{\pi}{3}:\quad \dfrac{dy}{dx}=\dfrac{\dfrac{3}{2}-3\left(\dfrac{1}{4}-\dfrac{3}{4}\right)}{-\dfrac{3\sqrt{3}}{2}+\dfrac{3\sqrt{3}}{2}}=$ undefined

$\theta=\pi:\quad \dfrac{dy}{dx}=\dfrac{-3-3\left(1-0\right)}{0+0}=$ undefined

$\theta=\dfrac{2\pi}{3}:\quad \dfrac{dy}{dx}=\dfrac{-\dfrac{3}{2}-3\left(\dfrac{1}{4}-\dfrac{3}{4}\right)}{-\dfrac{3\sqrt{3}}{2}-\dfrac{3\sqrt{3}}{2}}=0$

$\theta=\dfrac{3\pi}{2}:\quad \dfrac{dy}{dx}=\dfrac{0-3(0-1)}{-3(-1)+0}=1$

43) $\int_0^{2\pi} \frac{1}{2}\left(4+2\cos\theta\right)^2 d\theta = 18\pi \approx 56.549$ (graphing calculator used)

Just for kicks and giggles, let's do it "the hard way":

$$\int_0^{2\pi} \frac{1}{2}\left(16+16\cos\theta+4\cos^2\theta\right)d\theta = \int_0^{2\pi} \left(8+8\cos\theta+2\cos^2\theta\right)d\theta$$

$$= \left(8\theta+8\sin\theta\right)\Big|_0^{2\pi} + 2\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 16\pi + \int_0^{2\pi} \left(1+\cos 2\theta\right)d\theta = 16\pi + \left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = 16\pi + 2\pi = 18\pi$$

44)

$$\int_0^{2\pi} \frac{1}{2}\left(2+2\sin\theta\right)^2 d\theta = \int_0^{2\pi} \left(2+2\sin\theta+2\sin^2\theta\right)d\theta$$

$$= \left(2\theta-2\cos\theta\right)\Big|_0^{2\pi} + 2\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= 4\pi - 2 - \left(0-2\right) + \int_0^{2\pi} \left(1-\cos 2\theta\right)d\theta = 4\pi + \left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = 4\pi + 2\pi = 6\pi$$

45) $\int_0^{\pi/2} \frac{1}{2}\left(\cos 2\theta\right)^2 d\theta = \frac{\pi}{8}$

46) $8\int_0^{\pi/4} \frac{1}{2}\left(2\sin 4\theta\right)^2 d\theta = 2\pi$

47) $2\int_0^{\pi/4} \frac{1}{2}\left(4\cos 2\theta\right)d\theta = 2$

48) $6\int_0^{\pi/3} \frac{1}{2}\left(2\sin 3\theta\right)d\theta = 4$

49) $\int_0^{2\pi} \frac{1}{2}\left(3-2\cos\theta\right)^2 d\theta = 11\pi$

50) Inner loop is where $r = 0$: $2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Area inside inner loop: $\int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(2\sin\theta-1\right)^2 d\theta \approx 0.544$

51) Intersect (with respect to sin) at $r = 0$ and $r = \pi/4$

Area between:

$$2\int_0^{\pi/4} \frac{1}{2}\left(2\sin\theta\right)^2 d\theta = 2\int_0^{\pi/4} 2\sin^2\theta\,d\theta = 2\int_0^{\pi/4} \left(1-\cos 2\theta\right)d\theta$$

$$= 2\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = 2\left[\frac{\pi}{4}-\frac{1}{2}-\left(0-0\right)\right] = \frac{\pi}{2}-1$$

52) Intersect when $2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Area between:

$$\int_{\pi/6}^{5\pi/6} \frac{1}{2}(1)^2 d\theta + 2\int_0^{\pi/6} \frac{1}{2}(2\sin\theta)^2 d\theta = \frac{\theta}{2}\Big|_{\pi/6}^{5\pi/6} + 2\int_0^{\pi/6} 2\sin^2\theta d\theta$$

$$= \frac{\pi}{3} + 2\int_0^{\pi/6} (1-\cos 2\theta) d\theta = \frac{\pi}{3} + 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6} = \frac{\pi}{3} + 2\left[\frac{\pi}{6} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - (0-0)\right]$$

$$= \frac{\pi}{3} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

53) Intersect when $2 - 2\cos\theta = 2 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Area between:

$$\int_{\pi/2}^{3\pi/2} \frac{1}{2}(2)^2 d\theta + 2\int_0^{\pi/2} \frac{1}{2}(2-2\cos\theta)^2 d\theta = 2\theta\Big|_{\pi/2}^{3\pi/2} + 2\int_0^{\pi/2} (2 - 4\cos\theta + 2\cos^2\theta) d\theta$$

$$= 2\pi + 2[2\theta - 4\sin\theta]_0^{\pi/2} + 2\int_0^{\pi/2} (1+\cos 2\theta) d\theta = 2\pi + 2[\pi - 4 - (0-0)] + 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= 4\pi - 8 + 2\left[\frac{\pi}{2} + 0 - (0-0)\right] = 5\pi - 8$$

54) Intersect when $2 + 2\cos\theta = 2 - 2\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Area between:

$$4\int_0^{\pi/2} \frac{1}{2}(2-2\cos\theta)^2 d\theta = 4\int_0^{\pi/2} (2 - 4\cos\theta + 2\cos^2\theta) d\theta$$

$$= 4[2\theta - 4\sin\theta]_0^{\pi/2} + 4\int_0^{\pi/2} (1+\cos 2\theta) d\theta = 4[\pi - 4 - (0-0)] + 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= 4\pi - 16 + 4\left[\frac{\pi}{2} + 0 - (0-0)\right] = 6\pi - 16$$

55) Intersect when $2 = 2 - 2\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi$

Area between:

$$\int_0^{\pi} 2 d\theta - \int_0^{\pi} \frac{1}{2}(2-2\sin\theta)^2 d\theta = 2\theta\Big|_0^{\pi} - \int_0^{\pi} (2 - 4\sin\theta + 2\sin^2\theta) d\theta$$

$$= 2\pi - [2\theta + 4\cos\theta]_0^{\pi} - \int_0^{\pi} (1-\cos 2\theta) d\theta = 2\pi - [2\pi - 4 - (0+4)] - \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= 8 - [\pi - 0 - (0-0)] = 8 - \pi$$

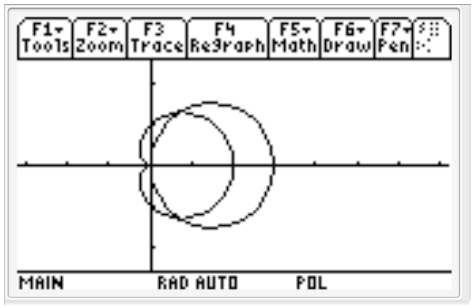
56) Intersect when $2 = 4\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi \Rightarrow \theta = \frac{\pi}{6} + n\pi, \frac{5\pi}{6} + n\pi, \frac{7\pi}{6} + n\pi, \frac{11\pi}{6} + n\pi$

Area between:

$$4\left[\int_{-\pi/6}^{\pi/6} \frac{1}{2}(4\cos 2\theta)^2 d\theta - \int_{-\pi/6}^{\pi/6} \frac{1}{2}(2)^2 d\theta\right] = 4\left[\int_{-\pi/6}^{\pi/6} (4(1+\cos 4\theta)) d\theta - 2\theta\Big|_{-\pi/6}^{\pi/6}\right]$$

$$= 4\left[[4\theta + \sin 4\theta]_{-\pi/6}^{\pi/6} - \frac{2\pi}{3}\right] = 4\left[\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) - \left(-\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \frac{2\pi}{3}\right] = 4\left[\frac{2\pi}{3} + \sqrt{3}\right] = \frac{8\pi}{3} + 4\sqrt{3}$$

57)



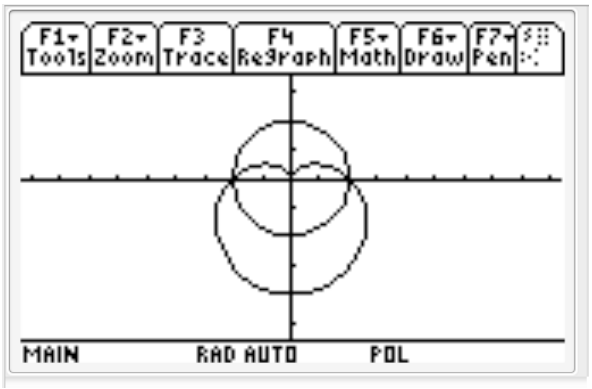
Intersect when $3\cos\theta = 1+\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}+2n\pi, \frac{5\pi}{3}+2n\pi$

Area between: $\int_{-\pi/3}^{\pi/3}\frac{1}{2}\left[\left(3\cos\theta\right)^2-\left(1+\cos\theta\right)^2\right]d\theta = \int_{-\pi/3}^{\pi/3}\frac{1}{2}\left[9\cos^2\theta-\left(1+2\cos\theta+\cos^2\theta\right)\right]d\theta$

$= \int_{-\pi/3}^{\pi/3}\frac{1}{2}\left[8\cos^2\theta-1-2\cos\theta\right]d\theta = \int_{-\pi/3}^{\pi/3}\frac{1}{2}\left[4\left(1+\cos2\theta\right)-1-2\cos\theta\right]d\theta$

$= \int_{-\pi/3}^{\pi/3}\left[\frac{3}{2}+2\cos2\theta-\cos\theta\right]d\theta = \left[\frac{3}{2}\theta+\sin2\theta-\sin\theta\right]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{2}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)-\left(-\frac{\pi}{2}-\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{3}}{2}\right)\right) = \pi$

58)



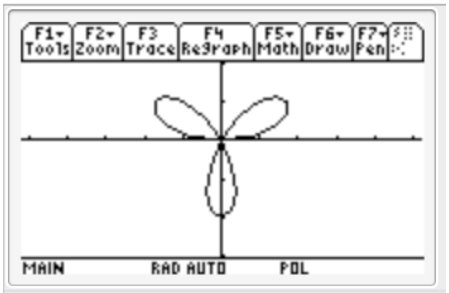
Intersect when $2 = 2-2\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi$

Area between: $\int_0^{\pi}\frac{1}{2}\left[\left(2\right)^2-\left(2-2\sin\theta\right)^2\right]d\theta = \int_0^{\pi}\frac{1}{2}\left[4-\left(4-8\sin\theta+4\sin^2\theta\right)\right]d\theta$

$= \int_0^{\pi}\frac{1}{2}\left[8\sin\theta-2\left(1-\cos2\theta\right)\right]d\theta = \int_0^{\pi}\frac{1}{2}\left[8\sin\theta-2+2\cos\theta\right]d\theta$

$= \int_0^{\pi}\left[4\sin\theta-1+\cos\theta\right]d\theta = \left[-4\cos\theta-\theta+\sin\theta\right]_0^{\pi} = \left(4-\pi+0\right)-\left(-4-0+0\right) = 8-\pi$

59)



Area enclosed: $\int_0^{\pi} \frac{1}{2}\left(2\sin 3\theta\right)^2 d\theta = \int_0^{\pi} 2\sin^2 3\theta\, d\theta = \int_0^{\pi} \left(1-\cos 6\theta\right) d\theta$

$$= \left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi} = \left(\pi - \frac{1}{6}(0)\right) - \left(0 - \frac{1}{6}(0)\right) = \pi$$

Slope at $\theta = \frac{\pi}{4} = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}$

$$x = 2\sin 3\theta(\cos\theta) \Rightarrow \frac{dx}{d\theta} = -2\sin 3\theta\sin\theta + 6\cos 3\theta\cos\theta$$

$$y = 2\sin 3\theta(\sin\theta) \Rightarrow \frac{dy}{d\theta} = 2\sin 3\theta\cos\theta + 6\cos 3\theta\sin\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{2\sin 3\theta\cos\theta + 6\cos 3\theta\sin\theta}{-2\sin 3\theta\sin\theta + 6\cos 3\theta\cos\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 6\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{-2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 6\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} = \frac{1-3}{-1-3} = \frac{1}{2}$$