Calculus Mid-Term Review *Focus: Chapter 3 Derivatives of a Function*

Concepts: Differentiability (do left- and right-hand derivatives equal?)
4 (actually 3) ways a function might fail to have a derivative
Intermediate Value Theorem for Derivatives
Rules for Differentiation ("shortcuts")
Power/Product/Quotient/Chain Rules
Second and Higher Order Derivatives
Position/Velocity/Acceleration Hierarchy and definition of Speed
Derivatives of Trigonometric Functions
Implicit Differentiation
Derivatives of Exponential and Logarithmic Functions

Problems:
Find all the values of x for which the function is differentiable.

1) $f(x) = \dfrac{3x^2 - 3x + 5}{x - 4}$ all reals except $x = 4$

2) $f(x) = \sqrt{x - 5}$ $(5, \infty)$

3) $f(x) = |3x - 5|$ all reals except $x = 5/3$

Find *dy/dx*.

4) $y = \dfrac{3x^2 - 3x + 5}{x - 4}$ $\dfrac{dy}{dx} = \dfrac{(x-4)(6x-3) - (3x^2 - 3x + 5)(1)}{(x-4)^2}$

5) $y = 3x^2 \cos(3x^2)$ $\dfrac{dy}{dx} = 3x^2(-\sin(3x^2))(6x) + 6x\cos(3x^2)$

6) $y = 3e^{x^2}x^2$ $\dfrac{dy}{dx} = 3e^{x^2}(2x) + 3e^{x^2}(2x)(x^2)$

7) $y = 3x^5 + 2x^4 + 8x^3 - 5x \ \ + 4$ $\dfrac{dy}{dx} = 15x^4 + 8x^3 + 24x^2 - 5$

8) $y = \sqrt{x^2 - 4x + 1}$ $\dfrac{dy}{dx} = \dfrac{1}{2}(x^2 - 4x + 1)^{-1/2}(2x - 4)$

9) $y = \dfrac{3\tan(x^2)}{2x}$ $\dfrac{dy}{dx} = \dfrac{2x(3\sec^2(x^2))(2x) - 3\tan(x^2)(2)}{4x^2}$

10) $y = \ln(\cos(3x^4))$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\cos\left(3x^4\right)}\left(-\sin\left(3x^4\right)\right)\left(12x^3\right) = -12x^3\tan\left(3x^4\right)$

11) $y = \log_4\left(3x^3\right)$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{3x^3\ln 4}\left(9x^2\right) = \dfrac{3}{x\ln 4}$

12) $y = 2^{\cos x^5}$ $\qquad \dfrac{dy}{dx} = 2^{\cos x^5}\ln 2\left(-\sin x^5\right)\left(5x^4\right)$

13) Find all of the nonzero derivatives of $y = 3x^5 + 2x^3 + 5x - 2$

$y' = 15x^4 + 6x^2 + 5$

$y'' = 60x^3 + 12x$

$y''' = 180x^2 + 12$

$y^{(4)} = 360x$

$y^{(5)} = 360$

Use implicit differentiation to find *dy/dx*:

14) $x^4 - y^4 = 3x + 2y$

$$4x^3 - 4y^3\frac{dy}{dx} = 3 + 2\frac{dy}{dx}$$

$$4x^3 - 3 = 2\frac{dy}{dx} + 4y^3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x^3 - 3}{2 + 4y^3}$$

15) $xy + 3x - 2y = 5$

$$x\frac{dy}{dx} + y + 3 - 2\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 2\frac{dy}{dx} = -y - 3$$

$$\frac{dy}{dx} = \frac{-y - 3}{x - 2}$$

16) $\cos x + \sin y = x^2y^3$

$$-\sin x + \cos y\left(\frac{dy}{dx}\right) = x^2\left(3y^2\frac{dy}{dx}\right) + 2xy^3$$

$$\cos y\left(\frac{dy}{dx}\right) - 3x^2y^2\frac{dy}{dx} = 2xy^3 + \sin x$$

$$\frac{dy}{dx} = \frac{2xy^3 + \sin x}{\cos y - 3x^2y^2}$$

Use implicit differentiation to find d^2y/dx^2:

17) $x^2 = 2 - \dfrac{3}{y}$

$$2x = 0 + \frac{3}{y^2}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2}{3}xy^2$$

$$\frac{d^2y}{dx^2} = \frac{2}{3}x\left(2y\frac{dy}{dx}\right) + \frac{2}{3}y^2$$

$$= \frac{4}{3}xy\left(\frac{2}{3}xy^2\right) + \frac{2}{3}y^2$$

$$= \frac{8}{9}x^2y^3 = \frac{2}{3}y^2$$

18) $x^2 - x = 3\sin y$

$$2x - 1 = 3\cos y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x-1}{3\cos y}$$

$$\frac{d^2y}{dx^2} = \frac{(3\cos y)(2) - (2x-1)(-3\sin y)\left(\frac{dy}{dx}\right)}{9\cos^2 y}$$

$$= \frac{(3\cos y)(2) - (2x-1)(-3\sin y)\left(\frac{2x-1}{3\cos y}\right)}{9\cos^2 y}$$

$$= \frac{6\cos y + (2x-1)^2 \tan y}{9\cos^2 y}$$

Calculus Mid-Term Review *Focus: Chapter 2 Limits and Continuity*

Concepts: Definition of a Limit
Properties of Limits
The Sandwich Theorem
Horizontal Asymptotes
Vertical Asymptotes
End Behavior Model
Continuity at a point
Intermediate Value Theorem for Continuous Functions
Tangent and Normal Lines to a curve

Problems:

1) $\lim_{x\to 1}\frac{x^2-x+1}{x+3}=\frac{1-1+1}{1+3}=\frac{1}{4}$

2) $\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$

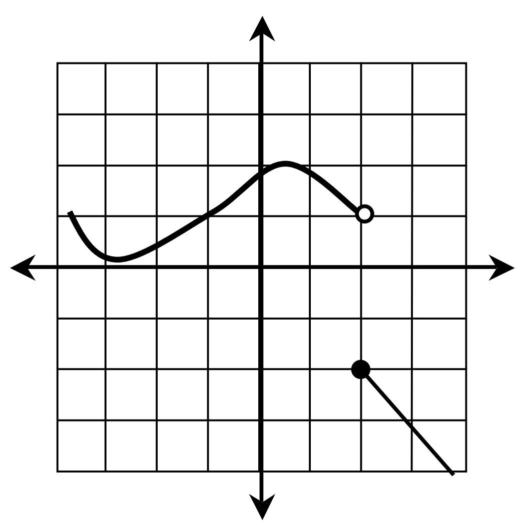
3) $\lim_{x\to 0}\frac{(2+x)^3-8}{x}=\lim_{x\to 0}\frac{8+12x+6x^2+x^3-8}{x}=\lim_{x\to 0}\left(12+6x+x^2\right)=12$

4) $\lim_{x\to 1}\frac{x^2-5}{x-1}=\lim_{x\to 1}\left(x+1+\frac{-4}{x-1}\right)=\infty$

5) $\lim_{x\to 1}\frac{x^2+3x-4}{x-1}\lim_{x\to 1}\frac{(x-1)(x+4)}{x-1}=\lim_{x\to 1}(x+4)=5$

6) $\lim_{x\to 0}\frac{3\sin^2 x-4\sin x}{\sin x}\lim_{x\to 0}\frac{\sin x(3\sin x-4)}{\sin x}=\lim_{x\to 0}(3\sin x-4)=-4$

7) Use the diagram to the right to answer the following:



A) $\lim_{x\to 2^-} f(x)$ = 1

B) $\lim_{x\to 2^+} f(x)$ = -2

C) $\lim_{x\to 2} f(x)$ = DNE

D) $f(2)$ = -2

Find the vertical asymptotes of the graph of $f(x)$. Then describe the behavior of $f(x)$ to the left and right of each vertical asymptote.

8) $f(x) = \dfrac{x^2 + 2x - 3}{x + 4}$ $x = -4$ $x \rightarrow -\infty$ from the left, ∞ from the right

9) $f(x) = 3x^2 + 2x$ no vertical asymptote

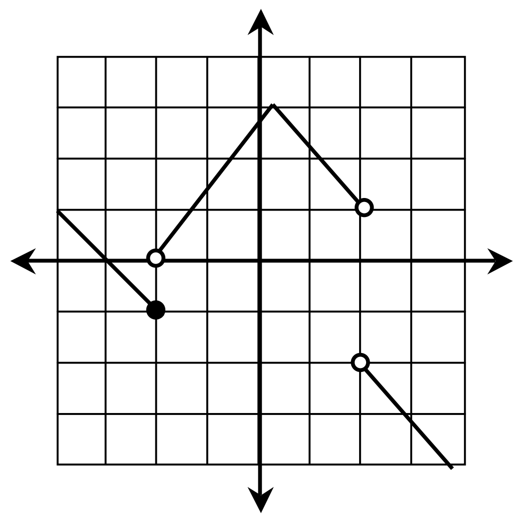
10) $f(x) = \dfrac{x+4}{x^2 + 2x - 3} = \dfrac{x+4}{(x+3)(x-1)}$ $x = -3$ $x \rightarrow \infty$ from the left, $-\infty$ from the right;

$x = 1$ $x \rightarrow -\infty$ from the left, ∞ from the right

11) For the functions in #8-10, find a power function end behavior model and identify any horizontal asymptotes.

For #8: end behavior model is $f(x) = x$, no horiz. asymptote.
For #9: end behavior model is $f(x) = 3x^2$, no horiz. asymptote.
For #10: end behavior model is $f(x) = \dfrac{1}{x}$, horiz. asymptote is $y = 0$.

12) Use the diagram to the right to answer the following:



A) $\lim_{x \to 2^-} f(x) = 1$

B) $\lim_{x \to 2^+} f(x) = -2$

C) $\lim_{x \to 2} f(x) =$ DNE

D) $f(2) =$ DNE

E) Is f continuous at $x = 2$? NO

F) $\lim_{x \to -2^-} f(x) = -1$

G) $\lim_{x \to -2^+} f(x) = 0$

H) $\lim_{x \to -2} f(x) =$ DNE

J) $f(-2) = -1$

K) Is f continuous at $x = -2$? NO

L) $\lim_{x \to -3^-} f(x) = 0$

M) $\lim_{x \to -3^+} f(x) = 0$

N) $\lim_{x \to -3} f(x) = 0$

P) $f(-3) = 0$

Q) Is f continuous at $x = -3$? YES

Find the average rate of change of the function over each interval.

13) $f(x)=3x^2+2x$ on [-2,3] $\qquad \dfrac{33-8}{3-(-2)}=\dfrac{25}{5}=5$

14) $f(x)=\cos(2x)$ on $\left[0,\dfrac{\pi}{3}\right]$ $\qquad \dfrac{-\frac{1}{2}-1}{\frac{\pi}{3}-0}=-\dfrac{9}{2\pi}$

15) $f(x)=\sqrt{x^2-4x+2}$ on [4,8] $\qquad \dfrac{\sqrt{34}-\sqrt{2}}{8-4}=\dfrac{\sqrt{34}-\sqrt{2}}{4}$

Use the definition that slope of a curve at a given point $x=a$ is $m=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ to find the slope of the given curve at the given point. Then find an equation for the tangent and normal lines to the curve at that point.

16) $f(x)=3x^2+2x$ at (1, 5)

$$m=\lim_{h\to 0}\frac{3(1+h)^2+2(1+h)-5}{h}=\lim_{h\to 0}\frac{3+6h+3h^2+2+2h-5}{h}=\lim_{h\to 0}(8+3h)=8$$

17) $f(x)=\dfrac{2}{x+2}$ at $\left(3,\dfrac{2}{5}\right)$

$$m=\lim_{h\to 0}\frac{3(1+h)^2+2(1+h)-5}{h}=\lim_{h\to 0}\frac{3+6h+3h^2+2+2h-5}{h}=\lim_{h\to 0}(8+3h)=8$$

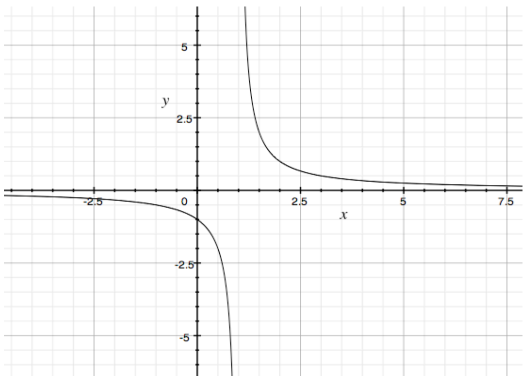
18) $f(x)=\sqrt{x-3}$ at $(7,2)$

$$m=\lim_{h\to 0}\frac{\sqrt{7+h-3}-\sqrt{7-3}}{h}=\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\bullet\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}=\lim_{h\to 0}\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)}=\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}=\frac{1}{4}$$

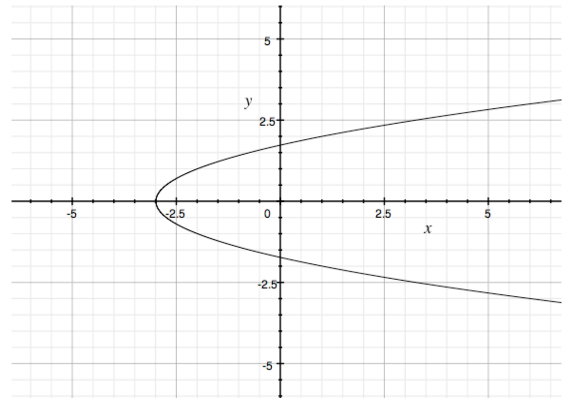
Calculus Mid-Term Review Name________________________________

Use the vertical line test to determine if the graph is a graph of a function. Write yes or no.

1) YES

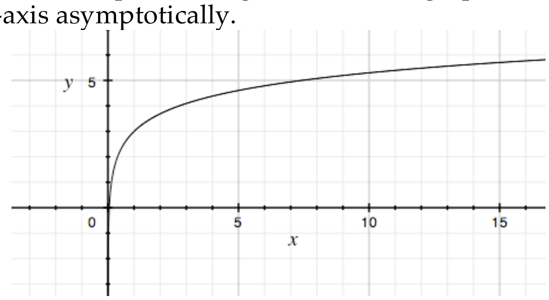


2) NO



Graph the exponential function.

3) $y = 3 + \ln x$ ln$x$ would pass through (1,0), so this graph will pass through (1,3). It will approach the y-axis asymptotically.



Find the formula for the function.

4) Express the area of an equilateral triangle with side lengths $5x$ as a function of the side length.

$$A = \frac{1}{2}bh = \frac{1}{2}(5x)\left(\frac{5x\sqrt{3}}{2}\right) = \frac{25x^2\sqrt{3}}{4}$$

Find the requested value.

5) Find $\lim\limits_{x\to4}\frac{x^2-16}{x-4} = \lim\limits_{x\to4}\frac{(x-4)(x+4)}{x-4} = \lim\limits_{x\to4}(x+4) = 8$

Find the exact value of the real number y.

6) $y = \cot^{-1}(-1) = -\frac{\pi}{4}$

Find the vertical asymptotes of the graph of $f(x)$.

7) $f(x) = \frac{x^2-16}{x-4}$ None. There is a removable discontinuity at $x = 4$

Solve the problem.

8) Find the points where the graph of the function has horizontal tangents.

$f(x) = 3 + \sqrt[3]{x}$ $\quad f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$, which is never equal to zero.

Find the average rate of change of the function over the given interval.

9) $f(x) = 3 + \sin x, \quad [0, \pi/2]$ $\quad \frac{f(b)-f(a)}{b-a} = \frac{(3+1)-(3+0)}{\frac{\pi}{2}-0} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

Find dy/dx by implicit differentiation. If applicable, express the result in terms of x and y.

10) $3y^3 - 5x^4 + 4 = 0$ $\quad 9y^2\frac{dy}{dx} - 20x^3 = 0 \Rightarrow \frac{dy}{dx} = \frac{20x^3}{9y^2}$

Find *dy/dx*.

11) $y=(\sin 4x)(x^2-3x+2)$

$$\frac{dy}{dx}=(\sin 4x)(2x-3)+(4\cos 4x)(x^2-3x+2)$$

Find the fourth derivative of the function.

12) $y = x^5 + 3x^4 + 2x^2 - 87$

$$y' = 5x^4 + 12x^3 + 4x$$

$$y'' = 20x^3 + 36x^2 + 4$$

$$y''' = 60x^2 + 72x$$

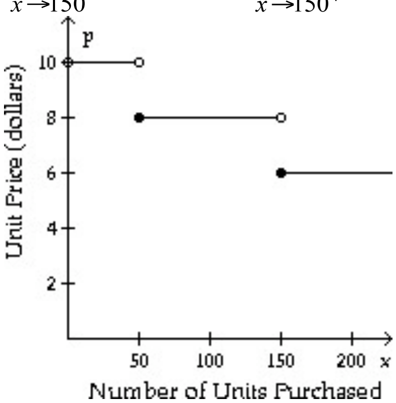
$$y^{(4)} = 120x + 72$$

Calculus Mid-Term Review page 2

Solve the problem.

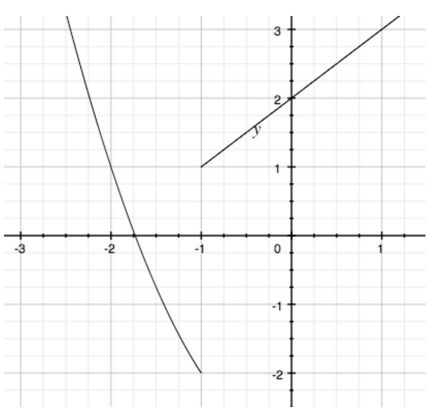
13) Suppose that the unit price, p, for x units of a product can be illustrated by the given graph. Find each limit, if it exists:

$$\lim_{x\to 150^-} p(x) = \mathbf{8},\ \lim_{x\to 150^+} p(x) = \mathbf{6},\ \lim_{x\to 150} p(x)\ \mathbf{DNE},\ \lim_{x\to 100} p(x) = \mathbf{8}$$



Determine the limit graphically, if it exists.

14) Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.



Suppose that the functions f and g and their derivatives with respect to x have the following values at the given values of x. Find the derivative with respect to x of the given combination at the given value of x.

15)

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	9	6	3
4	3	3	2	-5

$\dfrac{f(x)}{g(x)}$ at $x = 4$ $\qquad \dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2} = \dfrac{3(2) - 3(-5)}{3^2} = \dfrac{21}{9} = \dfrac{7}{3}$

Solve the problem.

16) A spherical balloon is being inflated at a rate of $40\pi\ \text{ft}^3/\text{min}$. ($V = \frac{4}{3}\pi r^3$, where r is the radius)

How fast is the radius of the balloon increasing when the radius is 1 ft?

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 40\pi = 4\pi\left(1^2\right)\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 10 \text{ ft/min}$$

Find the extreme values of the function on the interval and where they occur.

17) $f(x) = \cos\left(x - \dfrac{\pi}{4}\right), 0 \le x \le \dfrac{7\pi}{4}$

$f'(x) = -\sin\left(x - \dfrac{\pi}{4}\right) \qquad -\sin\left(x - \dfrac{\pi}{4}\right) = 0 \Rightarrow x - \dfrac{\pi}{4} = 0 \text{ or } x - \dfrac{\pi}{4} = \pi \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

The chart below shows that there is an absolute maximum of 1 at $x = \dfrac{\pi}{4}$, and absolute minimum of -1 at $x = \dfrac{5\pi}{4}$, a local minimum of $\dfrac{\sqrt{2}}{2}$ at $x = 0$, and a local maximum of 0 at $x = \dfrac{7\pi}{4}$

x	0	$\frac{\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
y	$\frac{\sqrt{2}}{2}$	1	-1	0

Find all points where the function is discontinuous.

18)
$$f(x)=\begin{cases}\dfrac{x^2-9}{x-3} & x\neq 3\\ 6 & x=3\end{cases}$$

There are NO points of discontinuity. $x = 3$ is a removable discontinuity, and $f(3) = \lim_{x\to 3} f(x)$.

Find the vertical asymptotes of the graph of $f(x)$.

19) $f(x) = \sec x$ Where $\cos x = 0$, or $\frac{\pi}{2} + n\pi$, where n is an integer.

Find the value of $(f \circ g)'$ at the given value of x.

20) $f(u) = \sin\frac{\pi u}{2}$, $u = g(x) = x^3$, $x = 1$

$$f'(x) = \left(\cos\frac{\pi x^3}{2}\right)\left(3x^2\right) \quad f'(1) = \left(\cos\frac{\pi 1^3}{2}\right)\left(3(1)^2\right) = (0)(3) = 0$$

Find the extreme values of the function on the interval and where they occur.

21) $f(x) = 3x^3 - 4; -4 \le x \le 5$ $\quad f'(x) = 9x^2 \quad 9x^2 = 0 \Rightarrow x = 0 \quad f''(x) = 9x$

The 2nd derivative test is inconclusive. $f(-4) = -196$; $f(0) = -4$; $f(5) = 371$. Abs. Min of -196 at $x = -4$, abs. max of 371 at $x = 5$

Find the critical values of the function.

22)
$$f(x) = 3x^4 + \frac{8}{3}x^3 - 2x^2 + 7$$
$$f'(x) = 12x^3 + 8x^2 - 4x = 4x(3x-1)(x+1)$$
$$4x(3x-1)(x+1) = 0 \Rightarrow x = -1, 0, \frac{1}{3}$$

Find the equation for the tangent to the curve at the given point.

23) $f(x) = 3x^2 - x^3$ at $(1,0)$

$$f'(x) = 6x - 3x^2 \quad f'(1) = 3 \quad y - 0 = 3(x-1) \Rightarrow y = 3x - 3$$

Find $f'(x)$ and state the domain of $f'(x)$.

24) $f(x) = \sqrt{9-x^2}$ $\quad f'(x) = \frac{1}{2}\left(9-x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{9-x^2}}$ Domain: (-3,3)

Use the Concavity Test to find the intervals where the graph of the function is concave up.

25) $y = 3x^3 + 18x^2 - 45x + 8$

$$y' = 9x^2 + 36x - 45$$
$$y'' = 18x + 36$$
$$18x + 36 = 0 \Rightarrow x = -2$$

Concave up on $(-2, \infty)$